University of Michigan
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EECS 569

PRODUCTION SYSTEMS ENGINEERING

Chapter 3: Mathematical Modeling of Production Systems

Instructor: Semyon M. Meerkov
GSI: Liang Zhang

- Motivation
  - All methods of PSE are model-based.
  - No two production systems are identical.
  - It is possible, however, to reduce “any” production systems to one of the small set of standard models.
  - These standard models and the process of reduction are the topics of this chapter.

OUTLINE

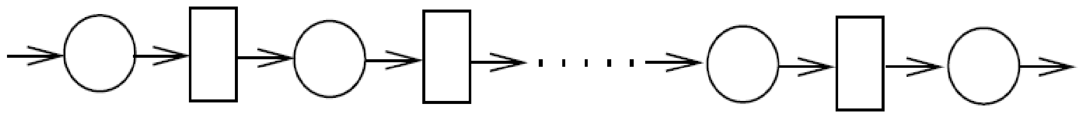
1. Types of Production Systems
2. Structural Modeling
3. Mathematical Models of Machines
4. Mathematical Models of Buffers
5. Modeling Interaction between Machines and Buffers
6. Performance Measures
7. Model Validation
8. Steps of Modeling, Analysis, Continuous Improvement, and Design
9. A Simplification: Transforming Exponential Models into Bernoulli Ones
10. Case Studies
11. Summary

1. Types of Production Systems

1.1 Serial Lines

1.1.1 Basic structure

- Consists of processing units and MHD arranged serially:

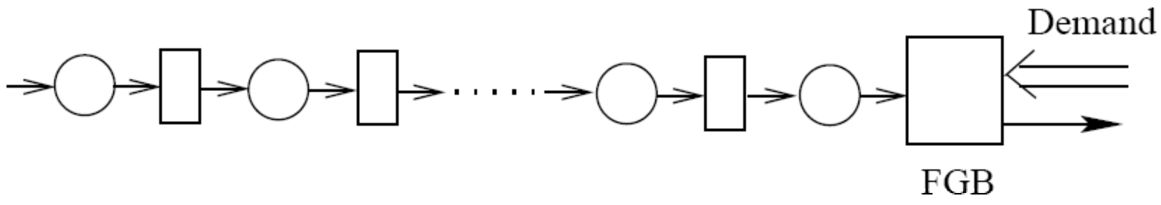


Processing unit: machine, manufacturing cell, shop in a plant, or a plant

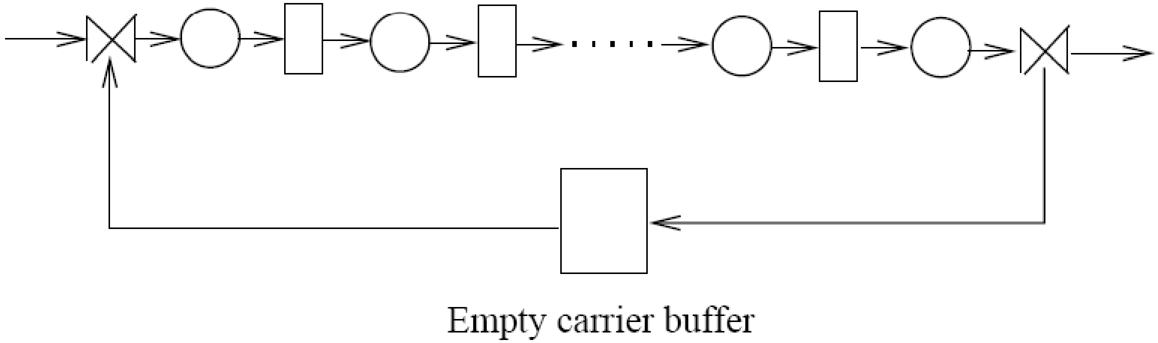
Material handling device: boxes, kanban cards, conveyors, AGVs, trucks, trains… The main feature: storing capacity.

1.1.2 Serial lines with FGB and carriers

- Serial lines with FGB:

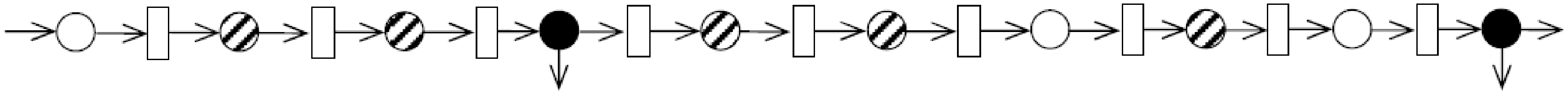


- Closed serial lines:

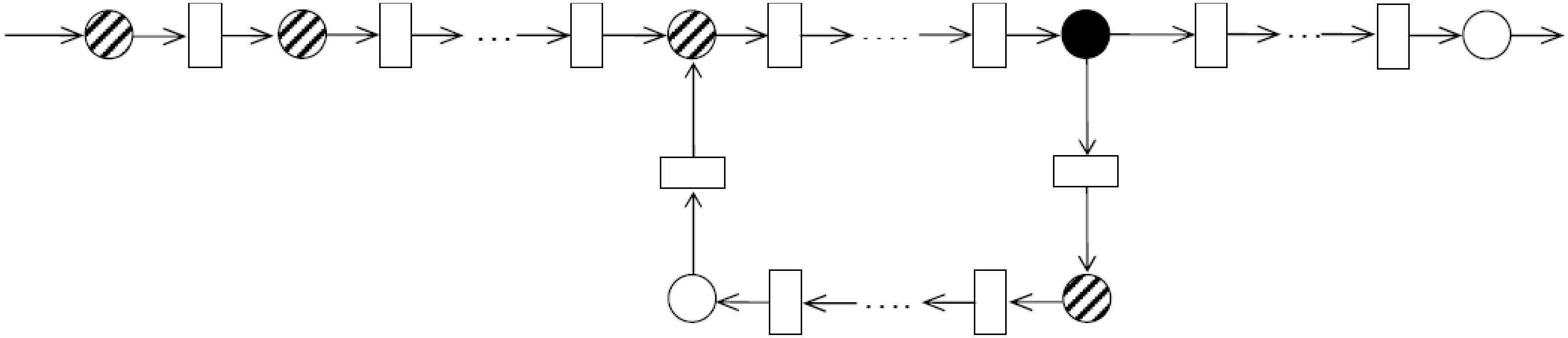


1.1.3 Serial lines with quality issues

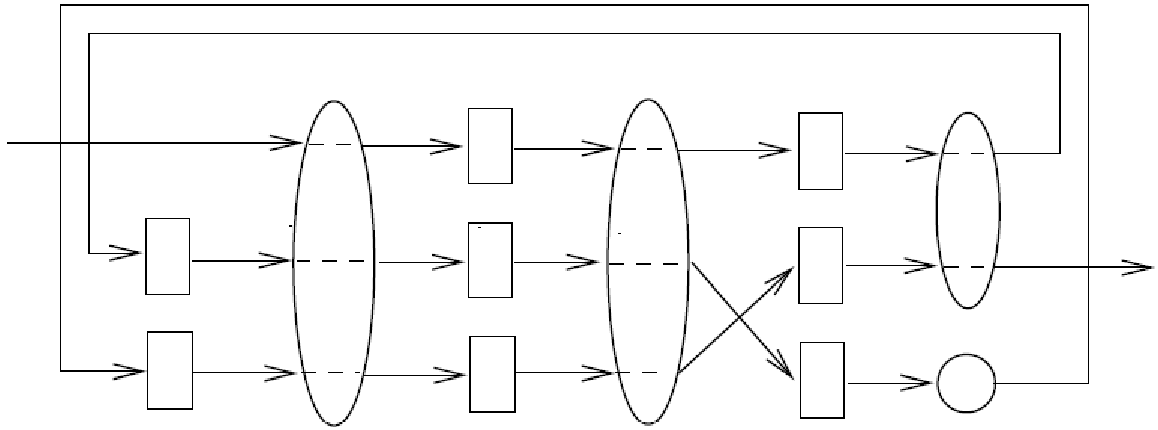
- Serial lines with inspection machines:



- Serial lines with rework:



1.1.4 Re-entrant lines

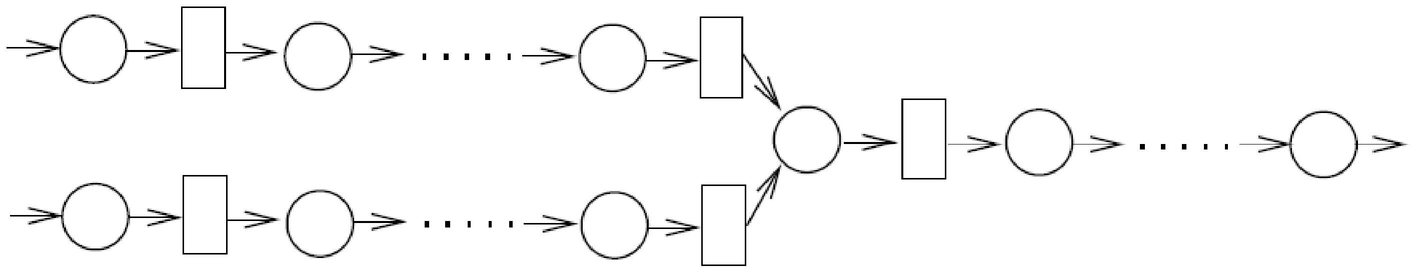


- Serial line is a work horse of manufacturing: every production system contains a serial line.
- Large part of PSE is devoted to these systems.

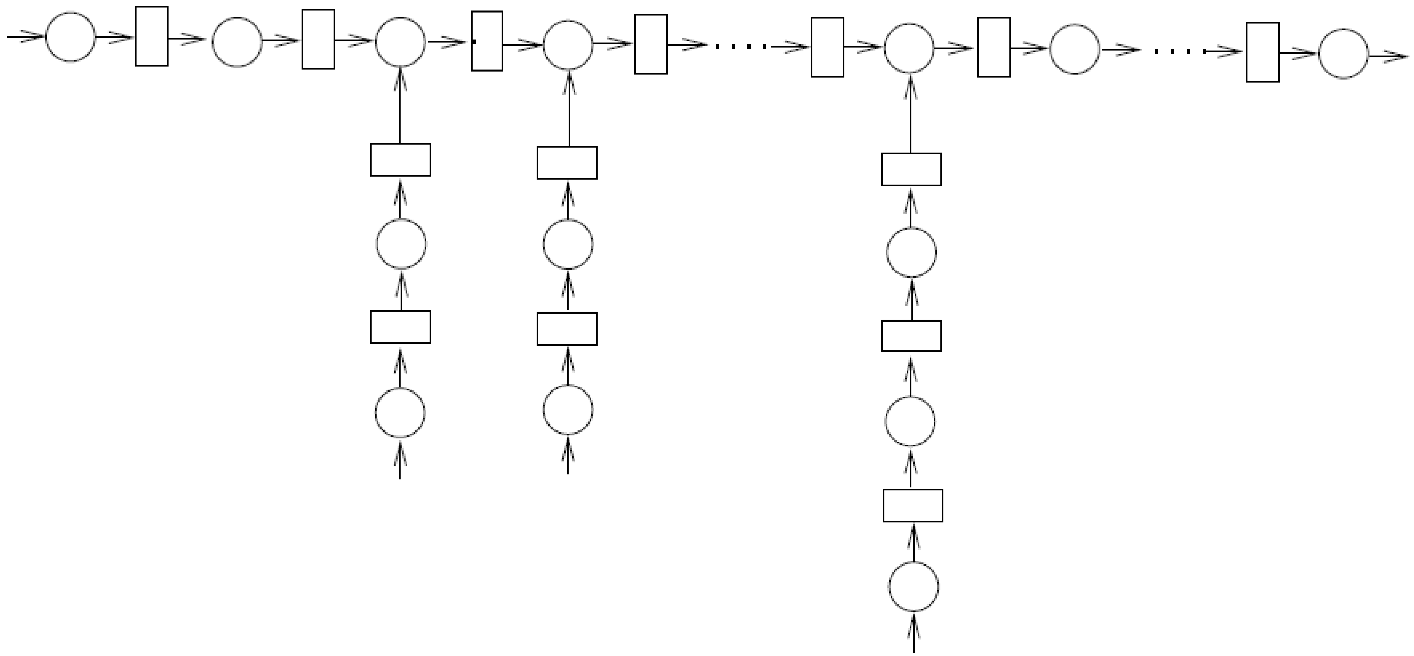
1.2 Assembly systems

1.2.1 Basic structure

- Two or more serial lines (component line) and one or more merge operations plus additional processing machines.



1.2.2 Assembly systems with multiple component lines and merge operations



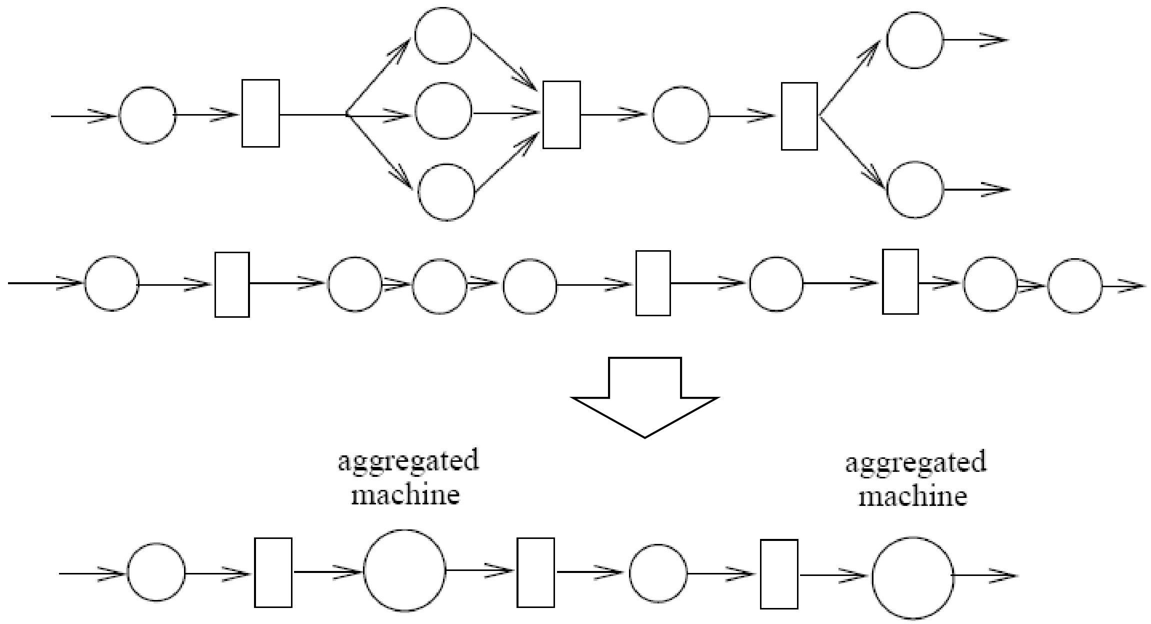
- All variations in structure are possible for assembly systems as well.

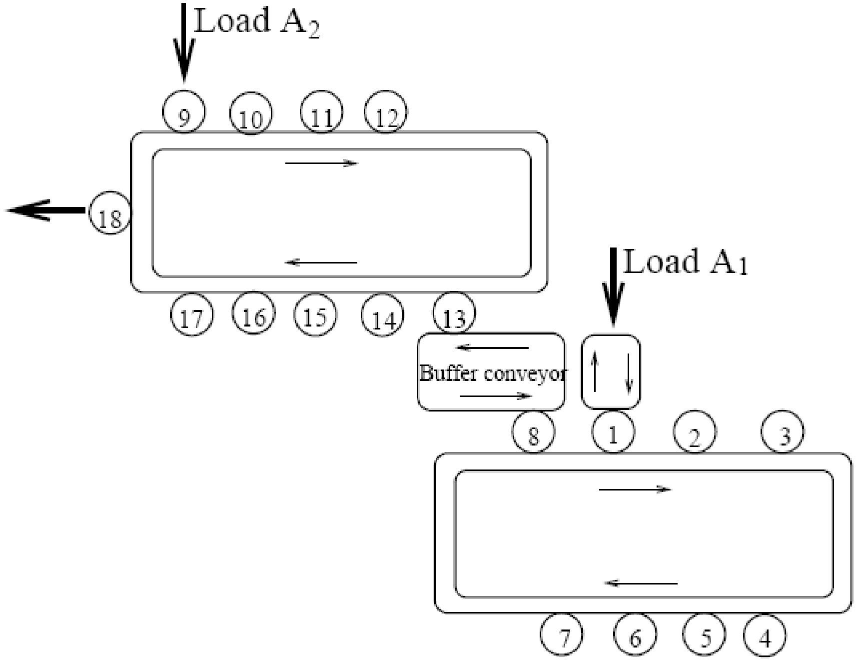
2. Structural Modeling

- Structural modeling – the process of reduction of a production system to one of the standard types.

  Example:

  - Serial lines with parallel or synchronous dependent machine.




Load A_2
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Buffer conveyor
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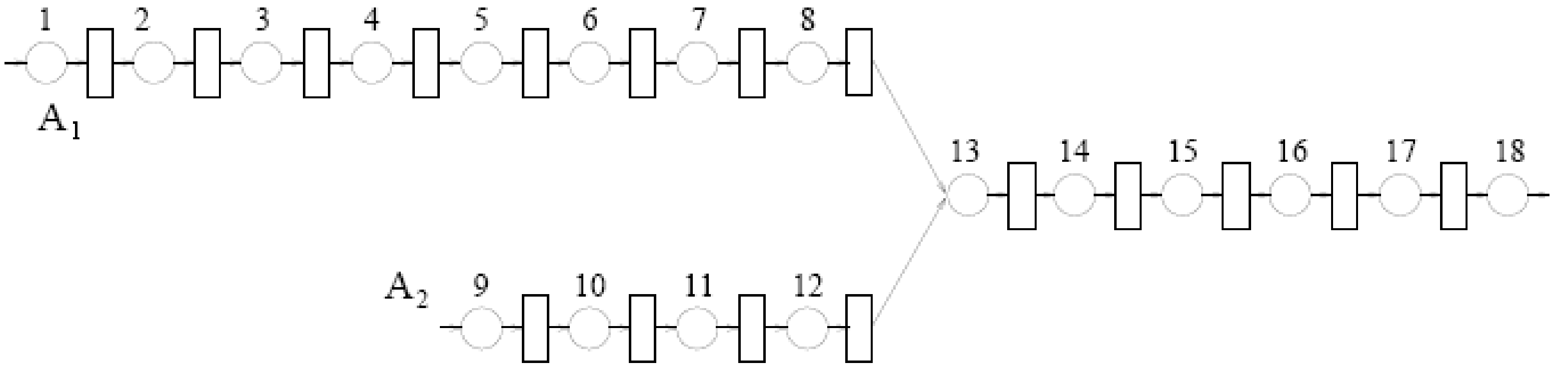
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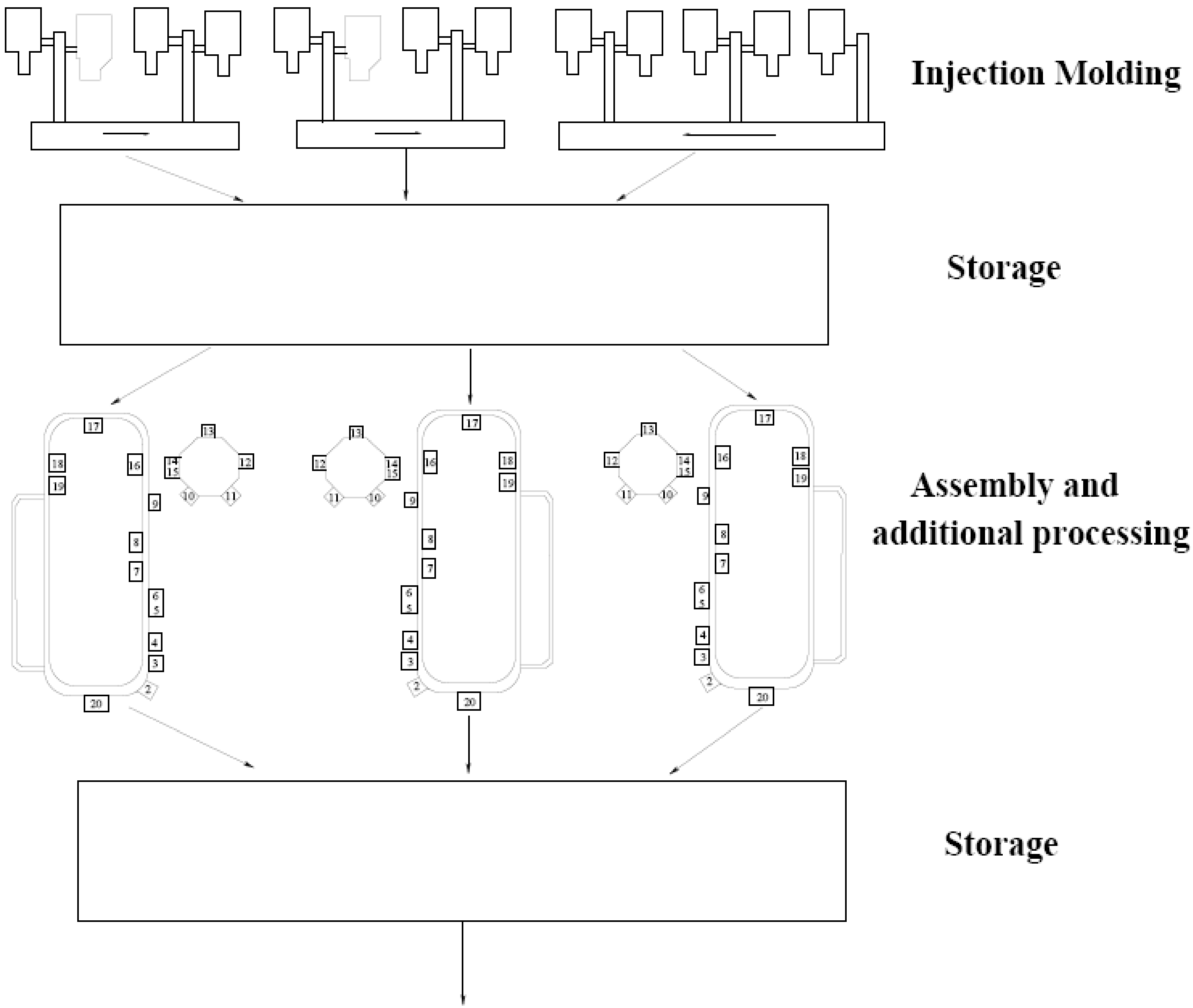
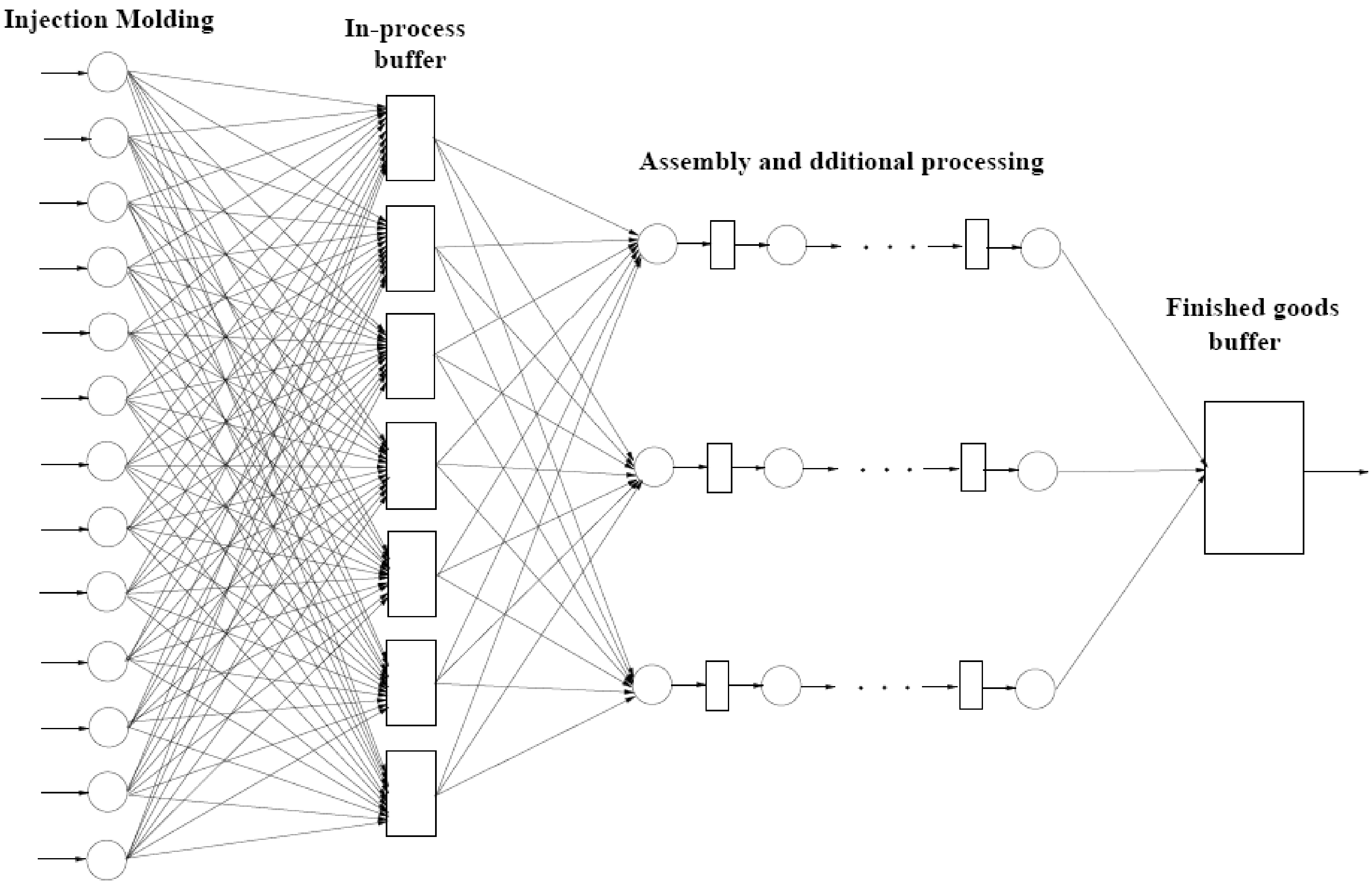
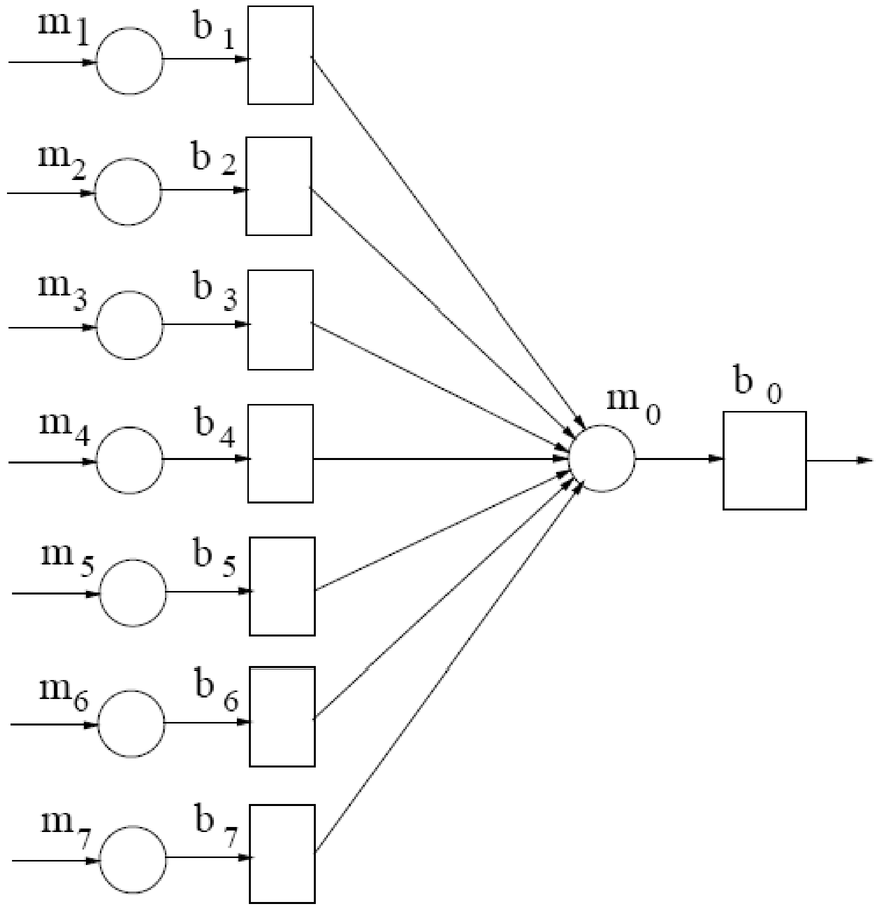

Injection Molding
Storage
Assembly and
additional processing
Storage


Injection Molding
In-process buffer
Assembly and dditional processing
Finished goods buffer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3. Mathematical Models of Machines

3.1 Timing issues

3.1.1 Machine cycle time and capacity

- *Cycle time* (τ) – time necessary to process a part by a machine.
  - τ may be constant, variable or random.
  - In large volume manufacturing it is mostly constant.

- *Machine capacity* (c) – number of parts produced per unit of time.

$$c = \frac{1}{\tau}$$

3.1.2 Slotted vs. unslotted time

- *Slotted time*: time is slotted with slot duration τ; all transitions occur at the beginning or the end of a slot.
  - Systems satisfying this property are called *synchronous*.
- *Unslotted* or *continuous time*: the above mentioned changes may occur at any time moment.
  - If the cycle times of all machines are identical, such a system is still referred to as *synchronous*.
  - If the cycle times are not identical, the system is called *asynchronous*.

3.1.3 Discrete event vs. flow systems

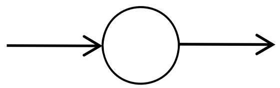
- *Discrete event systems*: only a complete job is transferred into or taken from a buffer.
- *Flow systems*: infinitesimal part of a job is transferred to or taken from a buffer.
- In the unslotted time case, flow systems are easier for analysis.

3.2 Machine reliability models

- *Machine reliability model* – pmf or pdf of up- and downtime

3.2.1 Reliability models for slotted time case:

- *Bernoulli* (B)



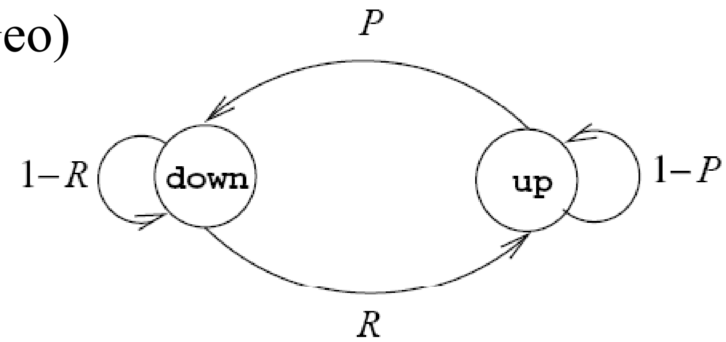
$$\begin{aligned} P[\{\text{machine is up during a cycle}\}] &= P[\{X=1\}] = p, \\ P[\{\text{machine is down during a cycle}\}] &= P[\{X=0\}] = 1-p. \end{aligned}$$

(static, no state)

Applicable to operations where downtime is comparable with cycle time.

3.2.1 Reliability models for slotted time case (cont.)

- *Geometric* (Geo)



Machine states: $\alpha(s) \in \{0, 1\}$

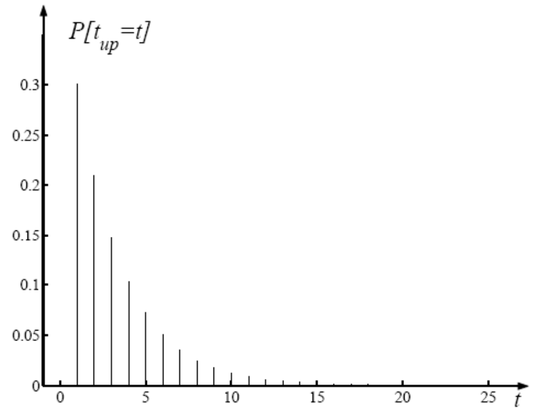
Transition probabilities:

$$P[\alpha(s+1) = 0 | \alpha(s) = 1] = P, \qquad P[\alpha(s+1) = 1 | \alpha(s) = 1] = 1 - P,$$
$$P[\alpha(s+1) = 1 | \alpha(s) = 0] = R, \qquad P[\alpha(s+1) = 0 | \alpha(s) = 0] = 1 - R.$$

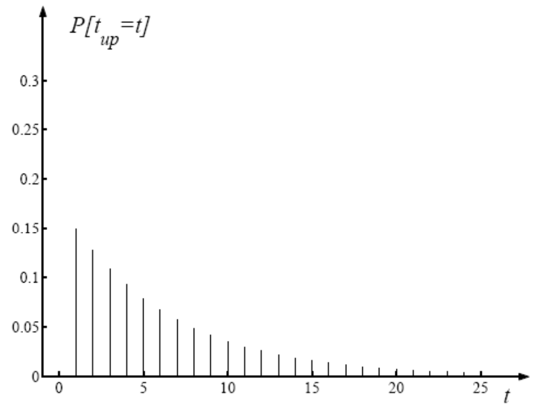
3.2.1 Reliability models for slotted time case (cont.)

- Resulting pmf's of up- and downtime:

$$P[t_{up} = t] = P(1-P)^{t-1}, \qquad t = 1, 2, \ldots$$
$$P[t_{down} = t] = R(1-R)^{t-1}, \qquad t = 1, 2, \ldots$$



(a) $P = 0.3$



(b) $P = 0.15$

- Memoryless pmf.
- Applicable when the downtime is much longer than the cycle time (e.g., some machining operations)

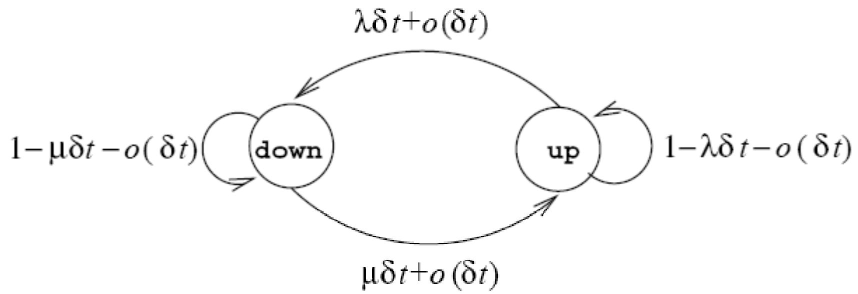
3.2.1 Reliability models for slotted time case (cont.)

- Moments of slotted time reliability models

Random variable	Expectation	Variance	CV
Bernoulli	p	$p(1-p)$	$\sqrt{\frac{1-p}{p}}$
Geometric	$\frac{1}{P}$	$\frac{1-P}{P^2}$	$\sqrt{1-P}$

3.2.2 Reliability models for continuous time case

- *Machines with a constant breakdown and repair rates*



$$
\begin{aligned}
P[\alpha(t+\delta t)=0|\alpha(t)=1] &= \lambda\delta t + o(\delta t) \\
P[\alpha(t+\delta t)=1|\alpha(t)=0] &= \mu\delta t + o(\delta t)
\end{aligned}
$$

$$\lim_{\delta t\to 0} \frac{o(\delta t)}{\delta t} = 0$$

3.2.2 Reliability models for continuous time case (cont.)

Calculation of uptime pdf's:

$$P[\alpha, t] = P[\{\text{machine is in state } \alpha \text{ at time } t\}], \qquad t \geq 0$$

Assume $P[1,0] = 1$. Then, by the total probability formula

$$\begin{aligned} P[1, t+\delta t] &= P[\alpha(t+\delta t) = 1 | \alpha(t) = 1] P[\alpha(t) = 1] \\ &\quad + P[\alpha(t+\delta t) = 1 | \alpha(t) = 0] P[\alpha(t) = 0] \end{aligned}$$

$$= (1 - \lambda \delta t - o(\delta t)) P[1, t]$$

3.2.2 Reliability models for continuous time case (cont)

$$\frac{P[1,t+\delta t]-P[1,t]}{\delta t}=-\lambda P[1,t]+\frac{o(\delta t)}{\delta t}$$

For $\delta t \to 0$,

$$\frac{dP[1,t]}{dt}=-\lambda P[1,t]$$

$$P[1,t]=e^{-\lambda t}, \qquad t \geq 0$$

3.2.2 Reliability models for continuous time case (cont)

The probability that the transition from up to down occurs during (t, $t+\delta t$):

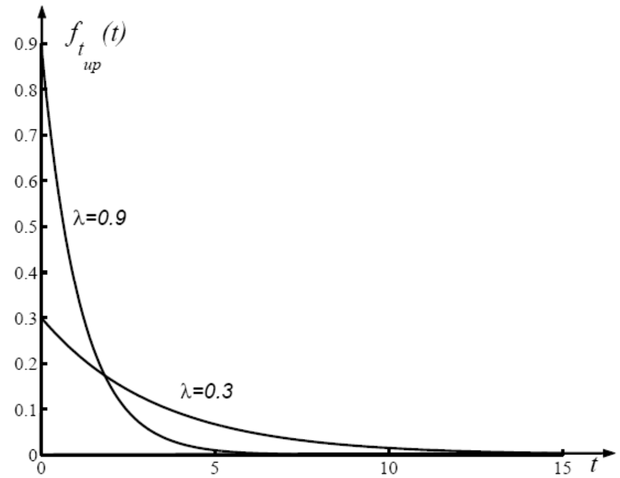
$$\begin{aligned} P[\{\alpha(t+\delta t)=0\}\cap\{\alpha(t)=1\}] &= P[\alpha(t+\delta t)=0|\alpha(t)=1\}]P[\alpha(t)=1] \\ &= (\lambda\delta t + o(\delta t))e^{-\lambda t} \\ &= \lambda e^{-\lambda t}\delta t + o(\delta t). \end{aligned}$$

Thus, a constant breakdown rate results in exponential pdf of the uptime:

$$f_{t_{up}}(t) = \lambda e^{-\lambda t}, \qquad t \geq 0.$$

A similar pdf (defined by the repair rate μ) is obtained for the downtime.

3.2.2 Reliability models for continuous time case (cont)

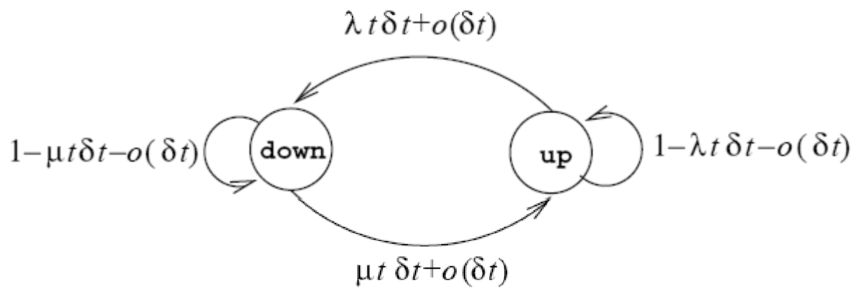


$$f_{t_{up}}(t) = \lambda e^{-\lambda t}, \qquad t \geq 0$$
$$f_{t_{down}}(t) = \mu e^{-\mu t}, \qquad t \geq 0$$

- Applicable to machining operations with a very small "most-probable" uptime and downtimes.

3.2.2 Reliability models for continuous time case (cont)

- *Machines with linearly increasing breakdown and repair rates.*

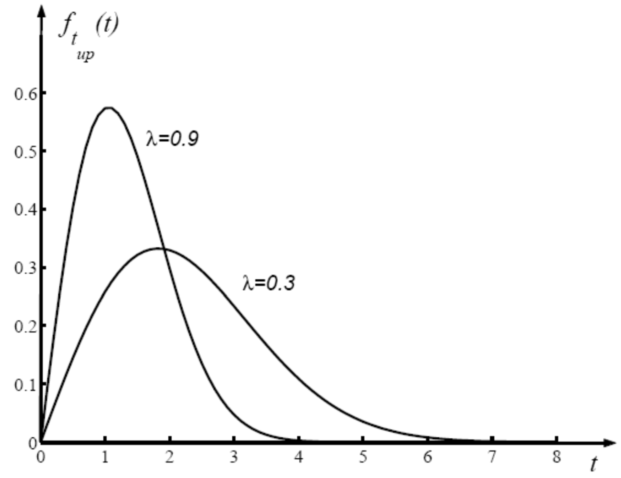


Using the same approach, we derive

$$f_{t_{up}}(t) = \lambda t e^{-\frac{\lambda t^2}{2}}, \qquad t \geq 0$$

which is the Rayleigh pdf.

3.2.2 Reliability models for continuous time case (cont)



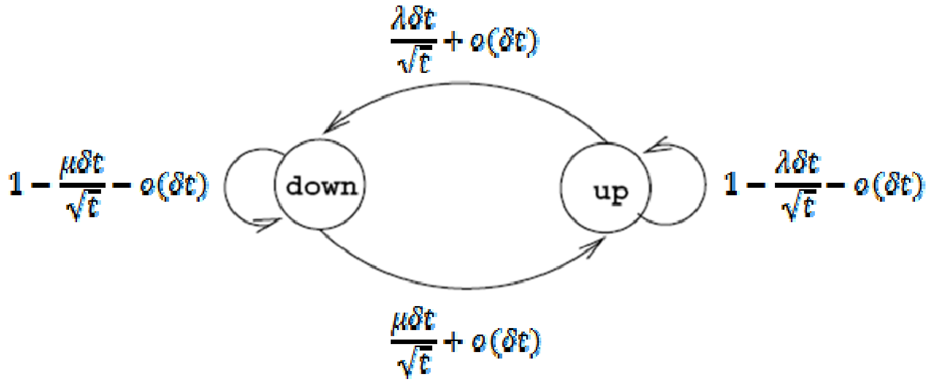
$$f_{t_{up}}(t) = \lambda t e^{-\frac{\lambda t^2}{2}}$$

$$f_{t_{down}}(t) = \mu t e^{-\frac{\mu t^2}{2}}$$

- Applicable to machining operations where “most-probable” up- and downtimes are away from 0.

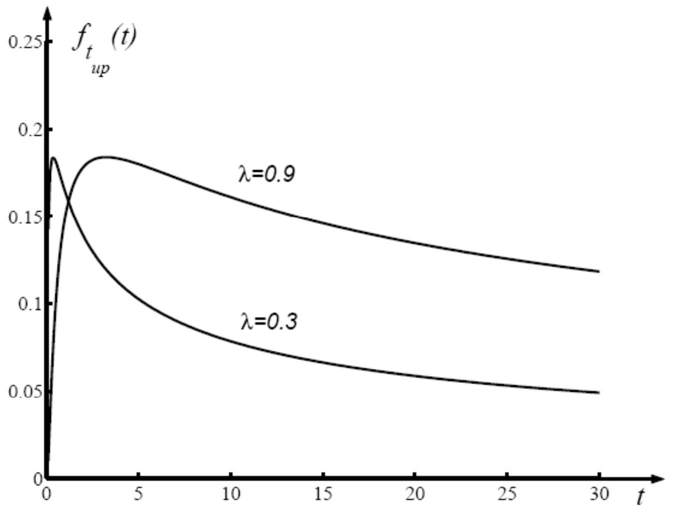
3.2.2 Reliability models for continuous time case (cont)

- *Machines with decreasing breakdown and repair rates*



$$f_{t_{up}}(t) = \frac{\lambda}{\sqrt{t}} e^{-\frac{2\lambda}{\sqrt{t}}}, \qquad t > 0$$

$$f_{t_{down}}(t) = \frac{\mu}{\sqrt{t}} e^{-\frac{2\mu}{\sqrt{t}}}, \qquad t > 0$$

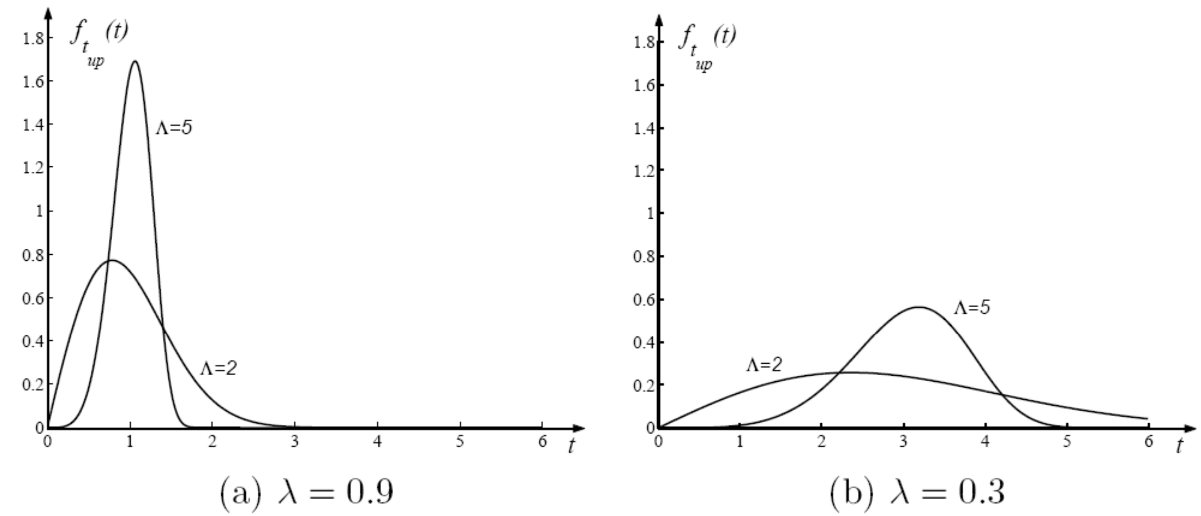


long tail

3.2.2 Reliability models for continuous time case (cont)

- *Weibull reliability model*

$$
\begin{aligned}
f_{t_{up}}(t) &= \lambda^{\Lambda} e^{-(\lambda t)^{\Lambda}} \Lambda t^{\Lambda-1}, \qquad t \geq 0, \\
f_{t_{down}}(t) &= \mu^{\mathrm{M}} e^{-(\mu t)^{\mathrm{M}}} \mathrm{M} t^{\mathrm{M}-1}, \qquad t \geq 0
\end{aligned}
$$



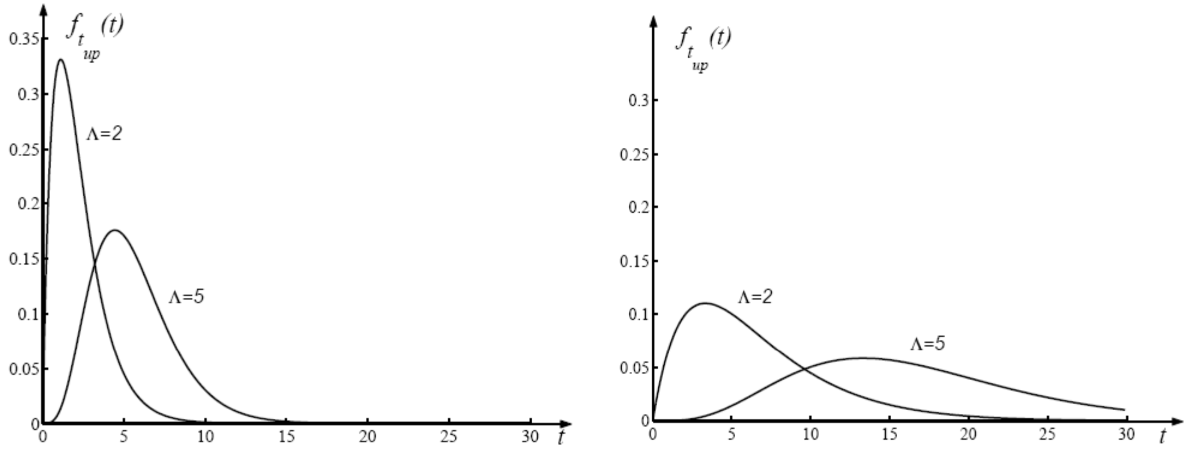
3.2.2 Reliability models for continuous time case (cont)

- *Gamma reliability model*

$$f_{t_{up}}(t) = \lambda e^{-\lambda t}\frac{(\lambda t)^{\Lambda-1}}{\Gamma(\Lambda)}, \qquad t \geq 0,$$

$$f_{t_{down}}(t) = \mu e^{-\mu t}\frac{(\mu t)^{\mathrm{M}-1}}{\Gamma(\mathrm{M})}, \qquad t \geq 0,$$

$$\Gamma(x) = \int_0^\infty s^{x-1}e^{-s}ds.$$



(a) $\lambda = 0.9$ (b) $\lambda = 0.3$

3.2.2 Reliability models for continuous time case (cont)

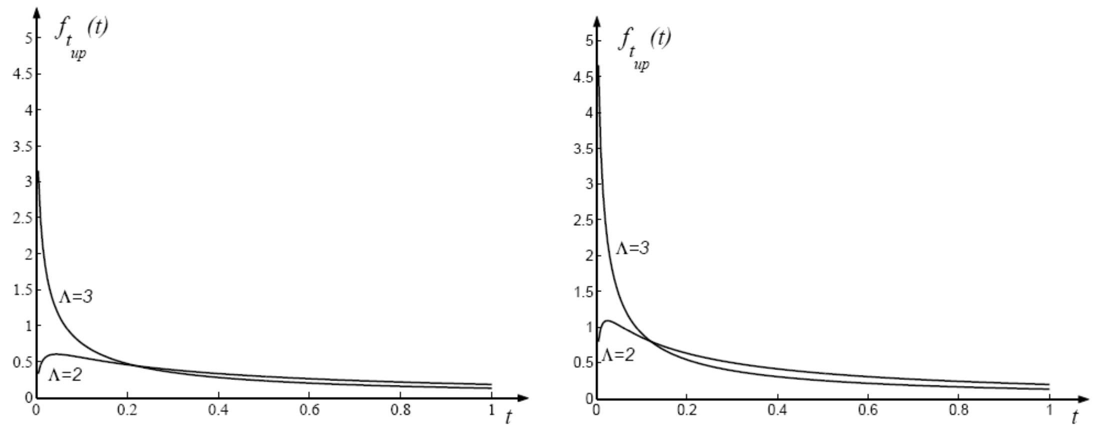
- *Log-normal reliability model*

Consider $t_{up} = e^X$, where *X* is a Gaussian random variable.

Then, t_{up} has the log-normal pdf:

$$f_{t_{up}}(t) = \frac{1}{\sqrt{2\pi}\Lambda t} e^{-\frac{(ln(t)-\lambda)^2}{2\Lambda^2}}, \qquad t > 0$$

$$f_{t_{down}}(t) = \frac{1}{\sqrt{2\pi}\mathrm{M}t} e^{-\frac{(ln(t)-\mu)^2}{2\mathrm{M}^2}}, \qquad t > 0$$



(a) $\lambda = 0.9$ (b) $\lambda = 0.3$

3.2.2 Reliability models for continuous time case (cont)

- *Mixed reliability model* (M): up- and downtime are distributed differently with pdf's from the set {exp, Ra, W, ga, LN}.

- *General reliability model* (G): up- and downtime have arbitrary distribution.

3.2.2 Reliability models for continuous time case (cont)

- Moments of continuous time reliability models

Random variable	Expectation	Variance	CV
Exponential	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	1
Rayleigh	$\sqrt{\frac{\pi}{2\lambda}}$	$\frac{4-\pi}{2\lambda}$	$\sqrt{\frac{4-\pi}{\pi}} = 0.523$
Decreasing transition rate	$\frac{1}{2\lambda^2}$	$\frac{5}{4\lambda^4}$	$\sqrt{5} = 2.236$
Gamma	$\frac{\Lambda}{\lambda}$	$\frac{\Lambda}{\lambda^2}$	$\frac{1}{\sqrt{\Lambda}}$
Weibull	$\frac{1}{\lambda}\Gamma(1+\frac{1}{\Lambda})$	$\frac{1}{\lambda^2}[\Gamma(1+\frac{2}{\Lambda}) - \Gamma^2(1+\frac{1}{\Lambda})]$	$\frac{\sqrt{\Gamma(1+\frac{2}{\Lambda})-\Gamma^2(1+\frac{1}{\Lambda})}}{\Gamma(1+\frac{1}{\Lambda})}$
Log-normal	$e^{\lambda+\frac{\Lambda^2}{2}}$	$e^{2\lambda+\Lambda^2}(e^{\Lambda^2}-1)$	$\sqrt{e^{\Lambda^2}-1}$

3.2.3 Efficiency of the machines (cont)

- For any reliability model, machine *efficiency* (e) is the probability to produce a part during a cycle time.

- Efficiency of machines with various reliability models are as follows:

  - Bernoulli: $e = p$

3.2.3 Efficiency of the machines (cont)

- Geometric

Markov balance equations:

$$\begin{aligned} P_1 &= P_{10}P_0 + P_{11}P_1 = RP_0 + (1-P)P_1, \\ P_0 &= P_{00}P_0 + P_{01}P_1 = (1-R)P_0 + PP_1, \end{aligned}$$
$$P_0 + P_1 = 1.$$

Steady state probabilities:

$$P_1 = \frac{R}{R+P}, \quad P_0 = \frac{P}{R+P}.$$

Since $T_{up} = \dfrac{1}{P}, \qquad T_{down} = \dfrac{1}{R},$

$$e = \frac{T_{up}}{T_{up} + T_{down}}$$

3.2.3 Efficiency of the machines (cont)

- Exponential

Markov balance equations:

$$\begin{aligned} \mu P_0 - \lambda P_1 &= 0, \\ P_0 + P_1 &= 1. \end{aligned}$$

Steady state probabilities:

$$\begin{aligned} P_1 &= \frac{\mu}{\mu + \lambda}, \\ P_0 &= \frac{\lambda}{\mu + \lambda}. \end{aligned}$$

Since $T_{up} = \frac{1}{\lambda}, \quad T_{down} = \frac{1}{\mu},$

$$e = \frac{T_{up}}{T_{up} + T_{down}}$$

3.2.3 Efficiency of the machines (cont)

- General

Let t_{up} and t_{down} be random up- and downtime distributed arbitrarily with expected values T_{up} and T_{down}. Assume that t_{up} and t_{down} are measured in units of cycle time.

Let

$$e_n = \frac{\sum_{i=1}^n t_{up,i}}{\sum_{i=1}^n (t_{up,i} + t_{down,i})} = \frac{\frac{1}{n}\sum_{i=1}^n t_{up,i}}{\frac{1}{n}\sum_{i=1}^n (t_{up,i} + t_{down,i})}.$$

Then

$$\begin{aligned}\lim_{n\to\infty} e_n &= \lim_{n\to\infty} \frac{\sum_{i=1}^n t_{up,i}}{\sum_{i=1}^n (t_{up,i} + t_{down,i})} \\ &= \frac{\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n t_{up,i}}{\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n t_{up,i} + \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n t_{down,i}}.\end{aligned}$$

3.2.3 Efficiency of the machines (cont)

- According to the strong law of large numbers, the following takes place with probability 1:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} t_{up,i} = T_{up},$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} t_{down,i} = T_{down}.$$

Thus,

$$e \overset{w.p.1}{=} \frac{T_{up}}{T_{up} + T_{down}}.$$

3.2.4 Coefficients of variations for up- and downtime of manufacturing equipment

- Manufacturing equipment on the factory floor often has coefficients of variation of up- and downtime less than 1.
- The explanation of this phenomenon is as follows:

- **Theorem:** If the breakdown (respectively, repair) rate is increasing in time, the resulting uptime (respectively, downtime) distribution has the coefficient of variation less than 1. If the breakdown (respectively, repair) rate is decreasing in time, the coefficient of variation is greater than 1.

3.2.5 Time dependent vs. operation-dependent failures

- *Time-dependent failures* – machine breakdown can occur even the machine is blocked or starved.
- *Operation-dependent failures* – machine breakdown occur only when machine is not blocked or starved.
- Both are practical. The difference in performance is small.
- In this course, we use time-dependent failures (easier to analyze).

3.3 Notations

- Machines:

$$[f_{t_{up}}(t), f_{t_{down}}(t)] \quad [\tau, f_{t_{up}}(t), f_{t_{down}}(t)]$$

$f_{t_{up}}(t), f_{t_{down}}(t)$ belong to $\{exp, Ra, W, ga, LN, G\}$

- Synchronous serial line:

$$\{[f_{t_{up}}(t), f_{t_{down}}(t)]_1, [f_{t_{up}}(t), f_{t_{down}}(t)]_2, \ldots, [f_{t_{up}}(t), f_{t_{down}}(t)]_M\}$$

$$\{[f_{t_{up}}(t), f_{t_{down}}(t)]_i, \quad i = 1, \ldots, M\}$$

- Asynchronous serial line:

$$\{[\tau, f_{t_{up}}(t), f_{t_{down}}(t)]_1, [\tau, f_{t_{up}}(t), f_{t_{down}}(t)]_2, \ldots, [\tau, f_{t_{up}}(t), f_{t_{down}}(t)]_M\}$$

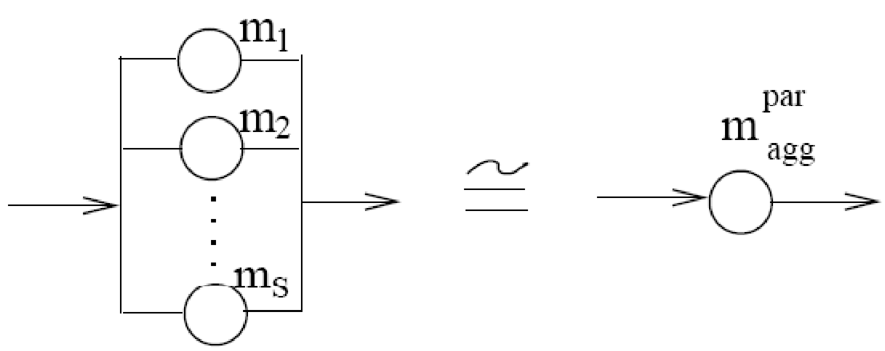
3.4 Machine model identification

- Goal: Determine $\tau, f_{t_{up}}(t), f_{t_{down}}(t)$
- τ: use “stop watch” (include loading/unloading time)
- $f_{t_{up}}(t)$ and $f_{t_{down}}(t)$ are practically impossible to identify.

3.4.1 What is possible (and necessary!) to identify

- $$T_{up} = \frac{\sum_{i=1}^{n} t_{up,i}}{n},$$
- $$T_{down} = \frac{\sum_{i=1}^{n} t_{down,i}}{n},$$
- $$Var(t_{up}) = \frac{\sum_{i=1}^{n} (t_{up,i} - T_{up})^2}{n-1},$$
- $$Var(t_{down}) = \frac{\sum_{i=1}^{n} (t_{down,i} - T_{down})^2}{n-1},$$
- $$CV_{up} = \frac{\sqrt{Var(t_{up})}}{T_{up}},$$
- $$CV_{down} = \frac{\sqrt{Var(t_{down})}}{T_{down}}.$$

3.4.2 Aggregating parallel machines for structural modeling



- Identical machines case:

$$m_i: \{\tau, T_{up}, T_{down}\}, i = 1,\ldots,S.$$

$$c_{agg}^{par} = Sc, \quad \tau_{agg}^{par} = \frac{1}{c_{agg}^{par}} = \frac{\tau}{S} \quad T_{up,agg}^{par} = T_{up}, \quad T_{down,agg}^{par} = T_{down}.$$

3.4.2 Aggregating parallel machines for structural modeling (cont)

- Non-identical machines case:

$$m_i: \{\tau_i, T_{up,i}, T_{down,i}\}, i = 1,\ldots,S.$$

⇩

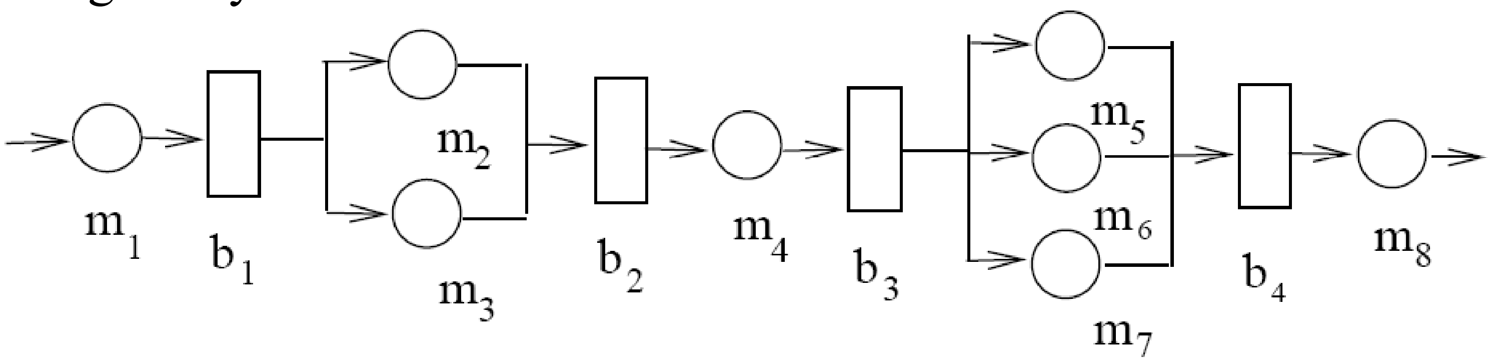
$$c_{agg}^{par} = \sum_{i=1}^{S} c_i, \qquad \tau_{agg}^{par} = \frac{1}{\sum_{i=1}^{S} \frac{1}{\tau_i}}$$

$$T_{up,agg}^{par} = \frac{\tau_{agg}^{par} \sum_{i=1}^{S} \left[\frac{1}{\tau_i T_{down,i}} \prod_{j=1,j\neq i}^{S} \left(\frac{1}{T_{up,j}} + \frac{1}{T_{down,j}}\right)\right]}{\frac{1}{S} \sum_{i=1}^{S} \left[\frac{1}{T_{up,i} T_{down,i}} \prod_{j=1,j\neq i}^{S} \left(\frac{1}{T_{up,j}} + \frac{1}{T_{down,j}}\right)\right]}$$

$$T_{down,agg}^{par} = \frac{\tau_{agg}^{par} \sum_{i=1}^{S} \left[\frac{1}{\tau_i T_{up,i}} \prod_{j=1,j\neq i}^{S} \left(\frac{1}{T_{up,j}} + \frac{1}{T_{down,j}}\right)\right]}{\frac{1}{S} \sum_{i=1}^{S} \left[\frac{1}{T_{up,i} T_{down,i}} \prod_{j=1,j\neq i}^{S} \left(\frac{1}{T_{up,j}} + \frac{1}{T_{down,j}}\right)\right]}$$

3.4.2 Aggregating parallel machines for structural modeling (cont)

- **Example**
  - Original system

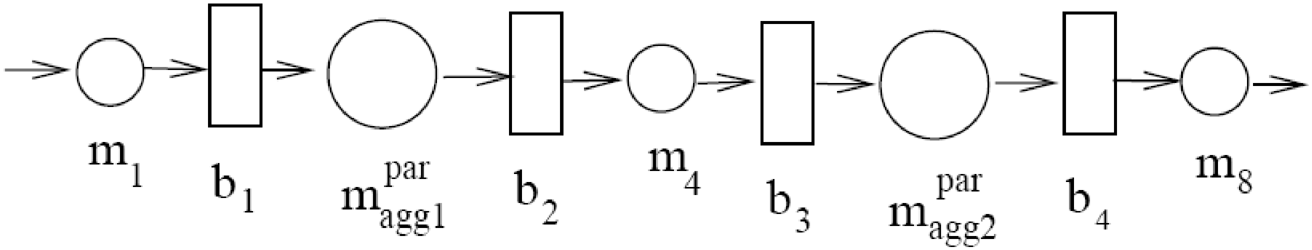


$$
\begin{aligned}
\tau &= \{1.2, 1.13, 1, 0.97, 1.4\}, \\
T_{down} &= \{10, 5.7091, 8.3333, 5.9039, 12.5\}, \\
T_{up} &= \{100, 40.8558, 20, 50.8103, 100\},
\end{aligned}
$$

$$
e = \{0.9091, 0.8774, 0.7059, 0.8959, 0.8889\}.
$$

3.4.2 Aggregating parallel machines for structural modeling (cont)

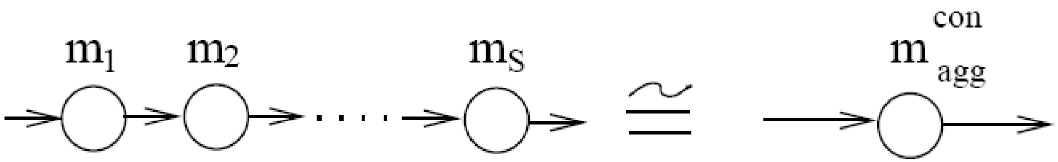
- Aggregated system



- Accuracy $\epsilon_{TP} = \frac{TP - TP_{agg}}{TP} \cdot 100\%$

N	TP	TP_{agg}	ϵ_{TP}
5	0.5388 ± 0.0002354	0.5105 ± 0.0002409	-5.25%
10	0.5874 ± 0.0002444	0.5699 ± 0.0002215	-2.98%
20	0.6202 ± 0.0001873	0.6138 ± 0.0002238	-1.03%
30	0.6292 ± 0.0002383	0.6272 ± 0.0002062	-0.32%
40	0.6326 ± 0.0001769	0.6318 ± 0.0002004	-0.13%
50	0.6339 ± 0.0002363	0.6332 ± 0.0002182	-0.11%
100	0.6347 ± 0.0001998	0.6346 ± 0.0002008	-0.02%

3.4.3 Aggregating consecutive dependent machines for structural modeling



- Identical machines case:

$$m_i\text{: } \{\tau, T_{up}, T_{down}\},\ i = 1,\ldots,S.$$

$$\begin{aligned}
\tau_{agg}^{con} &= \tau, \\
T_{up,agg}^{con} &= \left(\frac{T_{up}}{T_{up}+T_{down}}\right)^{S-1} T_{up}, \\
T_{down,agg}^{con} &= \left[1-\left(\frac{T_{up}}{T_{up}+T_{down}}\right)^{S}\right](T_{up}+T_{down}).
\end{aligned}$$

3.4.3 Aggregating consecutive dependent machines for structural modeling(cont.)

- Non-identical machines case:

$$m_i\text{: } \{\tau, T_{up}, T_{down}\},\ i = 1,\ldots,S.$$

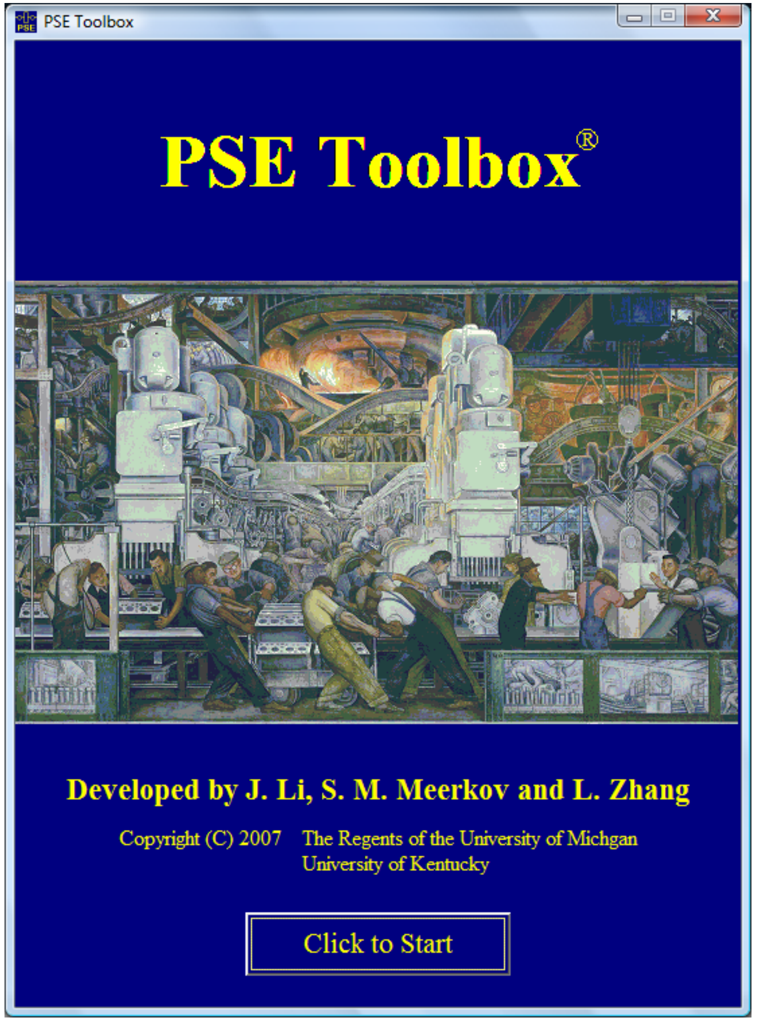
⇩

$$\tau_{agg}^{con} = \max_i \tau_i$$

$$T_{up,agg}^{con} = \frac{1}{S}\sum_{i=1}^{S}(T_{up,i} + T_{down,i})\prod_{i=1}^{S}\left(\frac{T_{up,i}}{T_{up,i} + T_{down,i}}\right),$$

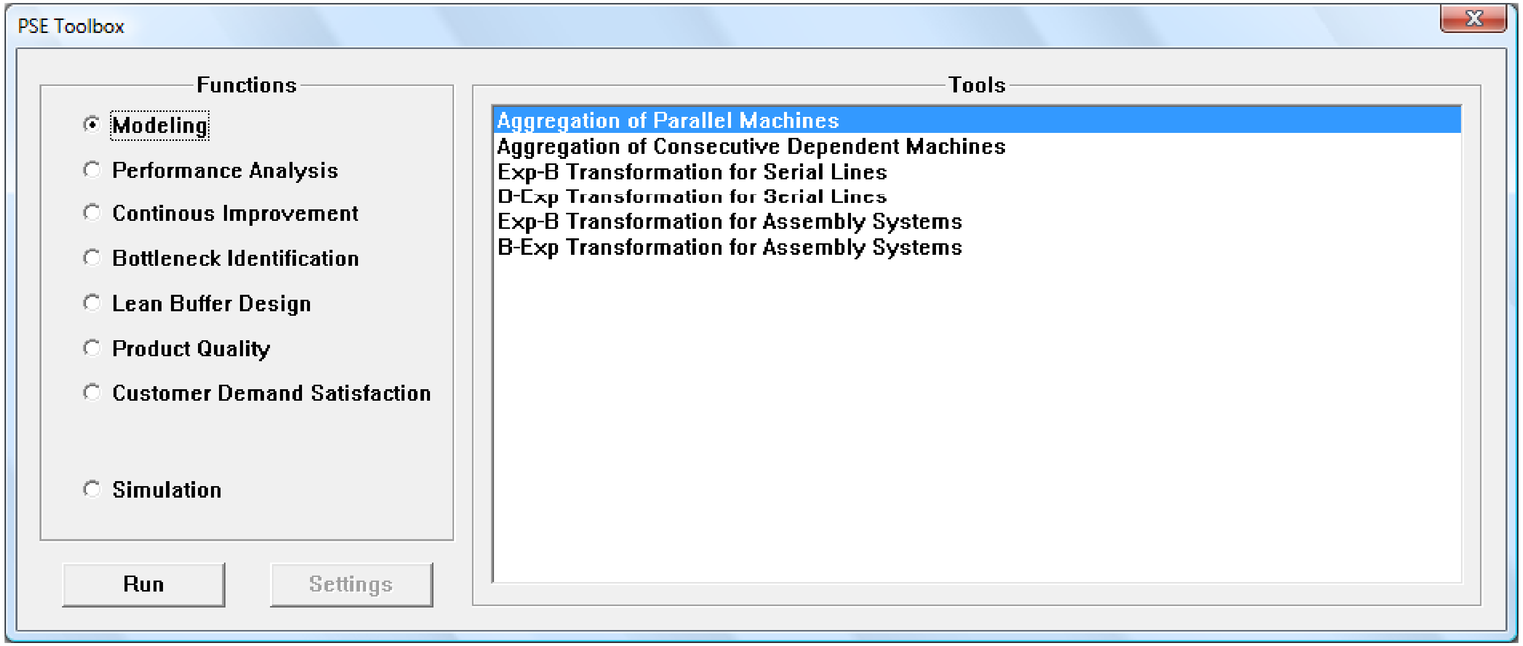
$$T_{down,agg}^{con} = \frac{1}{S}\sum_{i=1}^{S}(T_{up,i} + T_{down,i})\left[1 - \prod_{i=1}^{S}\left(\frac{T_{up,i}}{T_{up,i} + T_{down,i}}\right)\right].$$

3.4.4 PSE Toolbox



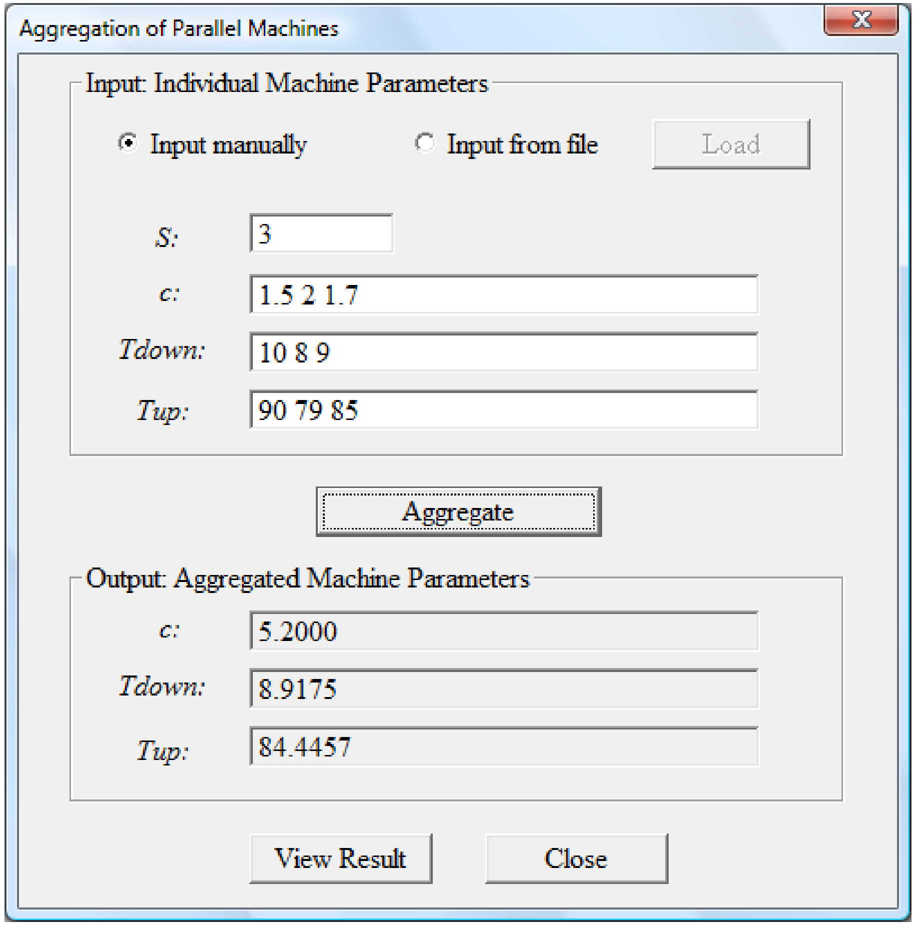
3.4.4 PSE Toolbox (cont.)

- **Modeling function and tools**



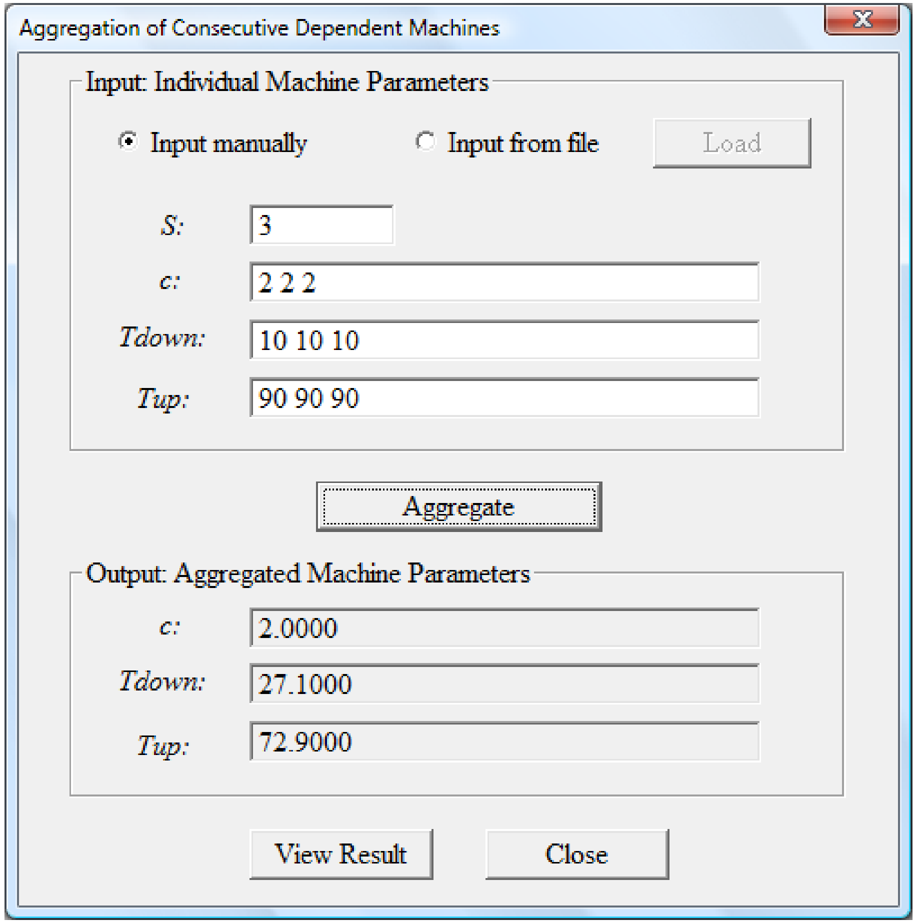
3.4.4 PSE Toolbox (cont.)

- **Aggregation of parallel machines**



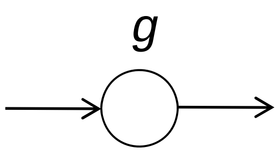
3.4.4 PSE Toolbox (cont.)

- Aggregation of consecutive dependent machines

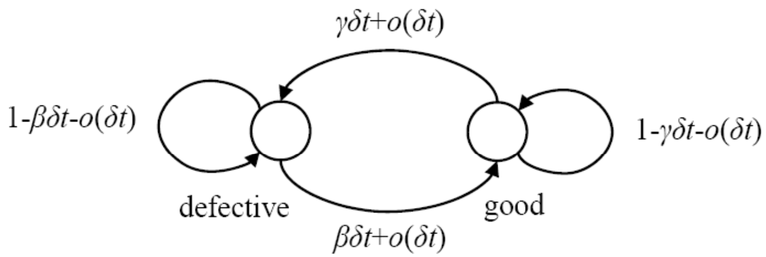


3.5 Machine quality models

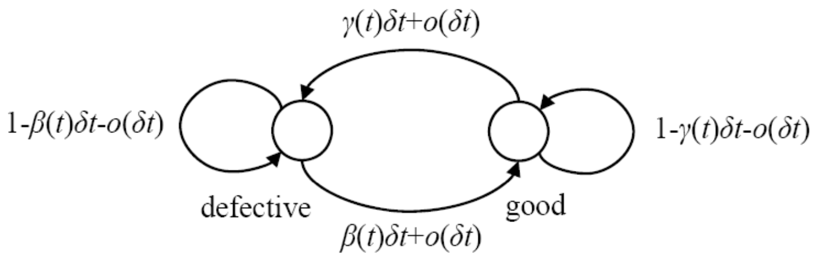
- *Machine quality model* – pmf or pdf of time intervals when defectives are produced.
  - *Bernoulli quality model*



  - *Exponential model*

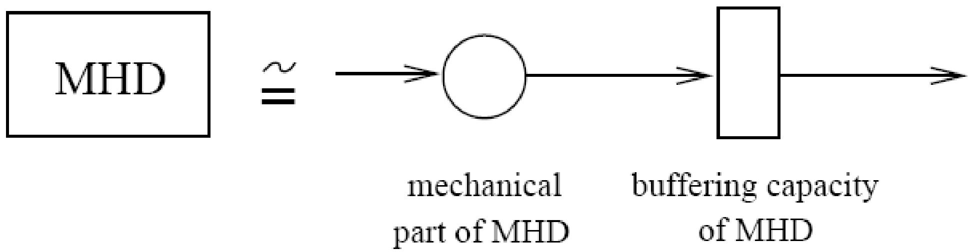


  - *General*



4. Mathematical Models of Buffers

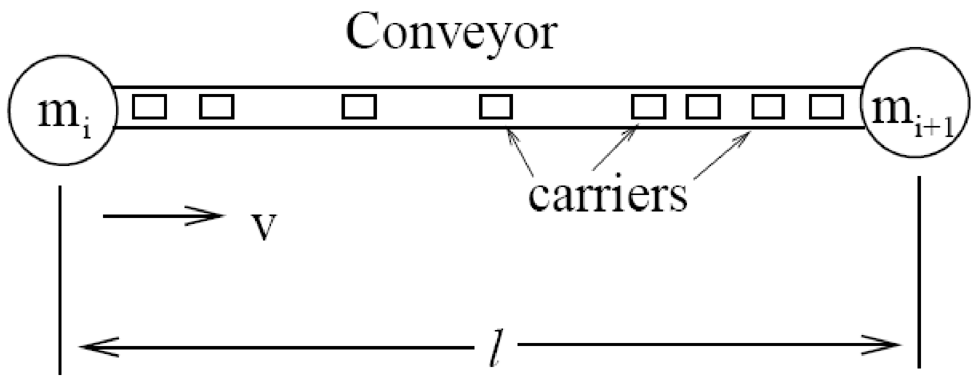
4.1 Structural model



- MHD identification: identification of $\{\tau, T_{up}, T_{down}\}$ of the machines part and identification of N.

- Often, mechanical part may be omitted.

4.2 Modeling conveyors



- $T_{travel} = \frac{l}{v}$
- Number of parts to sustain continuous operation

$$N_0 = \left\lfloor \frac{T_{travel}}{\tau} \right\rfloor$$

- Buffer capacity

$$N_i = K_i - N_0$$

K_i – number of parts that fit into the conveyor

5. Modeling Interaction between Machines and Buffers

5.1 Slotted time case

5.1.1 State changing conventions

- Machine state is determined at the beginning of each time slot.
- Buffer state is determined at the end of each time slot.

5.1.2 Blocking and starvation conventions

- *Blocked before service* (BBS) – a machine cannot operate if it is up, the downstream buffer is full and the downstream machine does not take a part from this buffer → The part in the machine is viewed as already in the buffer.
- *Blocked after service* (BAS) – if up and not starved, the machine operates even if the downstream buffer is full.

  This implies that

$$N_i^{BBS} = N_i^{BAS} + 1, \qquad i = 1, \ldots, M-1.$$

- *Starvation* – a machine is starved if it is up and the upstream buffer is empty.

6. Performance Measures

6.1 Production rate and throughput

- *Production rate* (*PR*) – average number of parts produced by the last machine per cycle time in the steady state.

⇩

Appropriate for synchronous systems.

- *Throughput* (*TP*) – average number of parts produced by the last machine per unit of time in the steady state.

⇩

Appropriate for both synchronous and asynchronous systems.

6.1 Production rate and throughput (cont)

- Functional dependence of *PR*:
  - For a Bernoulli line:

$$PR = PR(p_1, \ldots, p_M, N_1, \ldots, N_{M-1})$$

  - For a synchronous exponential line:

$$PR = PR(\lambda_1, \mu_1, \ldots, \lambda_M, \mu_M, N_1, \ldots, N_{M-1})$$

  - For an asynchronous exponential line:

$$TP = TP(\tau_1, \lambda_1, \mu_1, \ldots, \tau_M, \lambda_M, \mu_M, N_1, \ldots, N_{M-1})$$

  - For a synchronous line with Weibull machines:

$$PR = PR(\lambda_1, \Lambda_1, \mu_1, \mathrm{M}_1, \ldots, \lambda_M, \Lambda_M, \mu_M, \mathrm{M}_M, N_1, \ldots, N_{M-1})$$

  - For a closed Bernoulli line:

$$PR = PR(p_1, \ldots, p_M, N_1, \ldots, N_{M-1}, S, N_0)$$

6.1 Production rate and throughput (cont)

- For a Bernoulli assembly system:

$$\begin{aligned} PR &= PR(p_{11}, \ldots, p_{1M_1}, N_{11}, \ldots, N_{1M_1}; p_{21}, \ldots, p_{2M_2}, N_{21}, \ldots, N_{M_2}; \\ &\qquad p_{01}, \ldots, p_{0M_0}, N_{01}, \ldots, N_{0M_0-1}) \end{aligned}$$

- For a line with Bernoulli quality and reliability model:

$$PR = PR(p_1, \ldots, p_M, g_1, \ldots, g_M, N_1, \ldots, N_{M-1})$$

- None of these function can be evaluated in closed form for $M > 2$.
- For Bernoulli and exponential reliability models, these function can be evaluated analytically using recursive procedures.
- For other cases, empirical formulas can be derived.

6.2 Work-in-process and finished goods inventories

- WIP_i – average number of parts, i.e., *i*-th buffer, in the steady state.
- $WIP = \sum_{i=1}^{M-1} WIP_i$
- *Finished goods inventory* (FGI) – average number of parts in FGB in steady state.
- For Bernoulli lines:

$$WIP_i = WIP_i(p_1, \ldots, p_M, N_1, \ldots, N_{M-1})$$

- For exponential lines:

$$WIP_i = WIP_i(\lambda_1, \mu_1, \ldots, \lambda_M, \mu_M, N_1, \ldots, N_{M-1})$$

6.3 Probabilities of blockages and starvations

- *Blockage of* m_i (BL_i) – steady state probability that m_i is up, b_i is full and m_{i+1} does not take a part from b_i.
- *Starvation of* m_i (ST_i)– steady state probability that m_i is up and b_{i-1} is empty.
- For slotted time case:

$$\begin{aligned} BL_i &= P[\{m_i \text{ is up at the beginning of the time slot}\} \cap \{b_i \text{ is full at the} \\ &\quad \text{end of the previous time slot}\} \cap \{m_{i+1} \text{ does not take a part at} \\ &\quad \text{the beginning of the time slot}\}], \qquad i = 1, \ldots, M-1, \\ ST_i &= P[\{m_i \text{ is up at the beginning of the time slot}\} \cap \{b_{i-1} \text{ is empty at} \\ &\quad \text{the end of the previous time slot}\}], \qquad i = 2, \ldots, M. \end{aligned}$$

6.3 Probabilities of blockages and starvations (cont)

- For continuous time case:

$$
\begin{aligned}
BL_i &= P[\{m_i \text{ is up at time } t\} \cap \{b_i \text{ is full at time } t\} \cap \{m_{i+1} \text{ does not take} \\
&\qquad \text{material from } b_i \text{ at time } t\}], \qquad i = 1, \ldots, M-1, \\
ST_i &= P[\{m_i \text{ is up at time } t\} \cap \{b_{i-1} \text{ is empty at time } t\}], \quad i = 2, \ldots, M.
\end{aligned}
$$

- For exponential lines:

$$
\begin{aligned}
BL_i &= BL_i(\lambda_1, \mu_1, \ldots, \lambda_M, \mu_M, N_1, \ldots, N_{M-1}), \\
ST_i &= ST_i(\lambda_1, \mu_1, \ldots, \lambda_M, \mu_M, N_1, \ldots, N_{M-1}).
\end{aligned}
$$

- Closed form expressions are available only for Markovian systems with $M = 2$.
- For $M > 2$, recursive procedures are derived

6.4 Due-time performance

- *DTP* – the steady state probability to ship to the customer the desired number of parts, *D*, during a given shipping period, *T*:

$$DTP = P[\{\text{ship to the customer } D \text{ parts every shipping period } T\}]$$

- For exponential line

$$DTP = DTP(\lambda_1, \mu_1, \ldots, \lambda_M, \mu_M, N_1, \ldots, N_{M-1}, T, D, N_{FGB})$$

- Analytical expression is available only for $M = 1$.
- For $M > 1$, lower bound is available.

6.5 Transient characteristics

- *Transient characteristics* – a group of performance measures that describe how *PR* and *WIP* reach their steady states:
  - t_{sPR} – settling time for *PR*
  - t_{sWIP} – settling time for *WIP*
- All are characterized by the second largest eigenvalues of the transition matrix and pre-exponential coefficients.
- Solution are available for Markovian cases.

6.6 Measuring performance characteristics of the factory floor

- *PR* and *TP* are typically monitored.
- WIP_i less often.
- BL_i and ST_i very seldom – although they are most important for system "diagnosis".

7. Model Validation

- *Model validation* – the process of assessing accuracy of the mathematical model.
- Measures of accuracy:

$$\epsilon_{PR} = \frac{|PR - \widehat{PR}|}{PR} \cdot 100\%$$

$$\begin{aligned}
\epsilon_{WIP,i} &= \frac{|WIP_i - \widehat{WIP}_i|}{N_i} \cdot 100\%, \qquad i = 1, \ldots, M-1, \\
\epsilon_{ST,i} &= |ST_i - \widehat{ST}_i|, \qquad i = 2, \ldots, M, \\
\epsilon_{BL,i} &= |BL_i - \widehat{BL}_i|, \qquad i = 1, \ldots, M-1,
\end{aligned}$$

- Acceptable values.
  - Since the parameters of the machines and buffers are typically known with low accuracy (±5%), ε_{PR} within ±5% is typically acceptable.

8. Steps of Modeling, Analysis, Continuous Improvement, and Design

8.1 Modeling

- Layout investigation
- Structural modeling
- Machine parameter identification
- Buffer parameter identification
- Model validation

8.2 Analysis, continuous improvement, and design

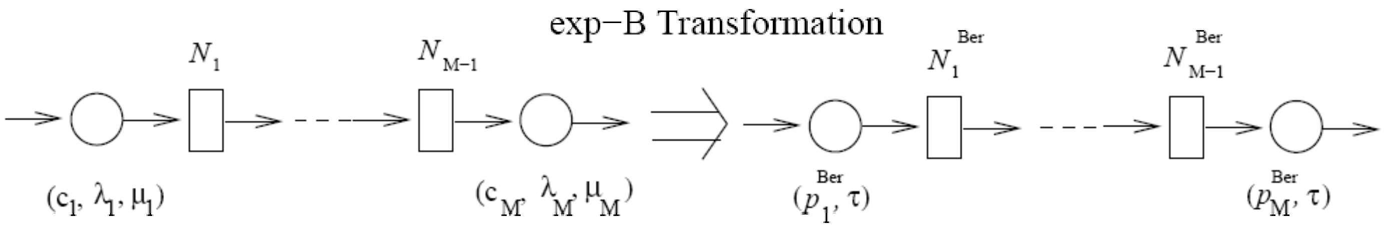
Using the mathematical model of a production system, the following problem can be solved:

- Evaluation of *PR*, *TP*, *WIP*, *BL*, *ST*, etc.
- Investigation of “what if” scenarios
- Direction for improvement

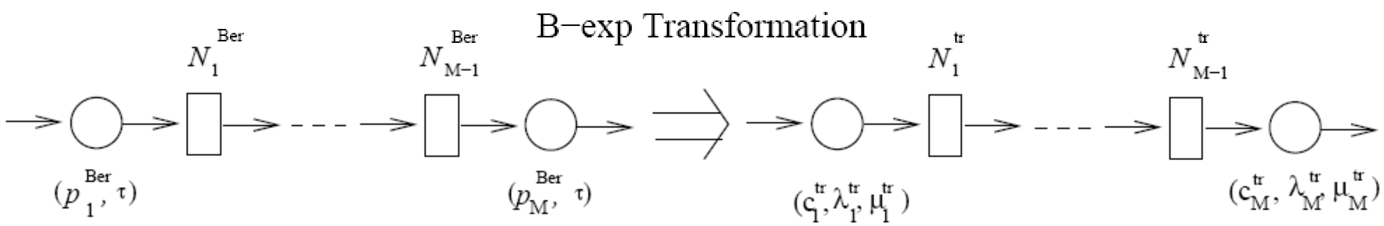
9. A Simplification: Transforming Exponential Models into Bernoulli Ones

9.1 Scenario

- Motivation – simplification
- exp-B transformation



- B-exp transformation



9.2 Exp-B transformation

- Expressions

$$
\begin{aligned}
\tau &= \frac{1}{c_{max}}, \\
p_i^{Ber} &= \frac{\frac{c_i}{c_{max}} \cdot \frac{1}{\lambda_i}}{\frac{1}{\lambda_i} + \frac{1}{\mu_i}} = \frac{c_i e_i}{c_{max}}, \qquad i = 1, \ldots, M, \\
N_i^{Ber} &= \min\left(\frac{N_i}{c_{i+1}}\mu_i, \frac{N_i}{c_i}\mu_{i+1}\right) + 1, \qquad i = 1, \ldots, M-1,
\end{aligned}
$$

$$
\begin{aligned}
c_{max} &= \max(c_i), \qquad i = 1, \ldots, M, \\
e_i &= \frac{\mu_i}{\lambda_i + \mu_i}, \qquad i = 1, \ldots, M.
\end{aligned}
$$

- **Accuracy** $\epsilon_{TP} = \frac{|TP^{exp} - TP^{Ber}|}{TP^{exp}} \cdot 100\%$

$M = 3, \lambda = 0.01, \mu = 0.1$

N_1	N_2	TP^{exp}	N_1^{Ber}	N_2^{Ber}	TP^{Ber}	$\frac{\lvert TP^{exp} - TP^{Ber}\rvert}{TP^{exp}} \cdot 100\%$
50	50	5.6234±0.0041	1.35	1.35	6.0895	8.3
50	100	5.7491±0.0045	1.35	1.7	6.1727	7.4
50	200	5.8768±0.0046	1.35	2.4	6.2396	6.2
50	400	5.9510±0.0045	1.35	3.8	6.2652	5.3
50	600	5.9641±0.0044	1.35	5.2	6.2677	5.1
100	50	5.8568±0.0041	1.7	1.35	6.2691	7
100	100	5.9811±0.0041	1.7	1.7	6.3565	6.3
100	200	6.1092±0.0043	1.7	2.4	6.4286	5.2
100	400	6.1863±0.0040	1.7	3.8	6.4581	4.4
100	600	6.2005±0.0039	1.7	5.2	6.4612	4.2
200	50	6.1323±0.0039	2.4	1.35	6.4701	5.5
200	100	6.2566±0.0038	2.4	1.7	6.564	4.9
200	200	6.3881±0.004	2.4	2.4	6.644	4
200	400	6.4701±0.0039	2.4	3.8	6.6794	3.2
200	600	6.4864±0.0037	2.4	5.2	6.6836	3
400	50	6.3715±0.0036	3.8	1.35	6.6288	4
400	100	6.5001±0.0035	3.8	1.7	6.7324	3.6
400	200	6.6392±0.0036	3.8	2.4	6.8234	2.8
400	400	6.7311±0.0033	3.8	3.8	6.8662	2
400	600	6.7517±0.0032	3.8	5.2	6.8719	1.8
600	50	6.466±0.0037	5.2	1.35	6.8719	3.3
600	100	6.5998±0.0035	5.2	1.7	6.7911	2.9
600	200	6.7462±0.0034	5.2	2.4	6.889	2.1
600	400	6.8454±0.0032	5.2	3.8	6.9362	1.3
600	600	6.8683±0.0031	5.2	5.2	6.9428	1.1

9.3 B-exp transformation

- Expressions

$$\mu_i^{tr} = \mu_i, \qquad i = 1, \ldots, M$$

$$\text{If } \frac{p_i^{Ber} c_{max}}{c_i} < 1, \text{ then } \; c_i^{tr} = c_i, \qquad \text{and} \qquad e_i^{tr} = \frac{p_i^{Ber} c_{max}}{c_i^{tr}}, \qquad i = 1, \ldots, M.$$

$$\text{If } \frac{p_i^{Ber} c_{max}}{c_i} \geq 1, \text{ choose } c_i^{tr} \text{ such that}$$

$$\frac{p_i^{Ber} c_{max}}{c_i^{tr}} < 1, \qquad \text{and} \qquad e_i^{tr} = \frac{p_i^{Ber} c_{max}}{c_i^{tr}}, \qquad i = 1, \ldots, M.$$

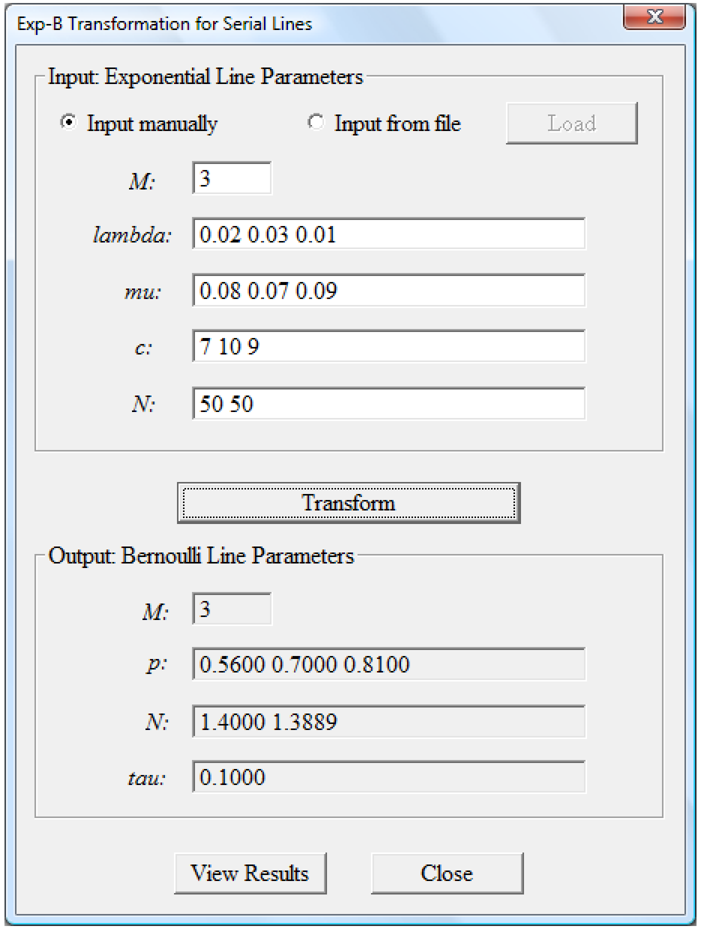
$$\lambda_i^{tr} = \frac{(1 - e_i^{tr})\mu_i^{tr}}{e_i^{tr}}$$

$$N_i^{tr} = \max\left(\frac{N_i^{Ber} - 1}{\mu_i^{tr}} c_{i+1}^{tr}, \frac{N_i^{Ber} - 1}{\mu_{i+1}^{tr}} c_i^{tr}\right), \qquad i = 1, \ldots, M - 1$$

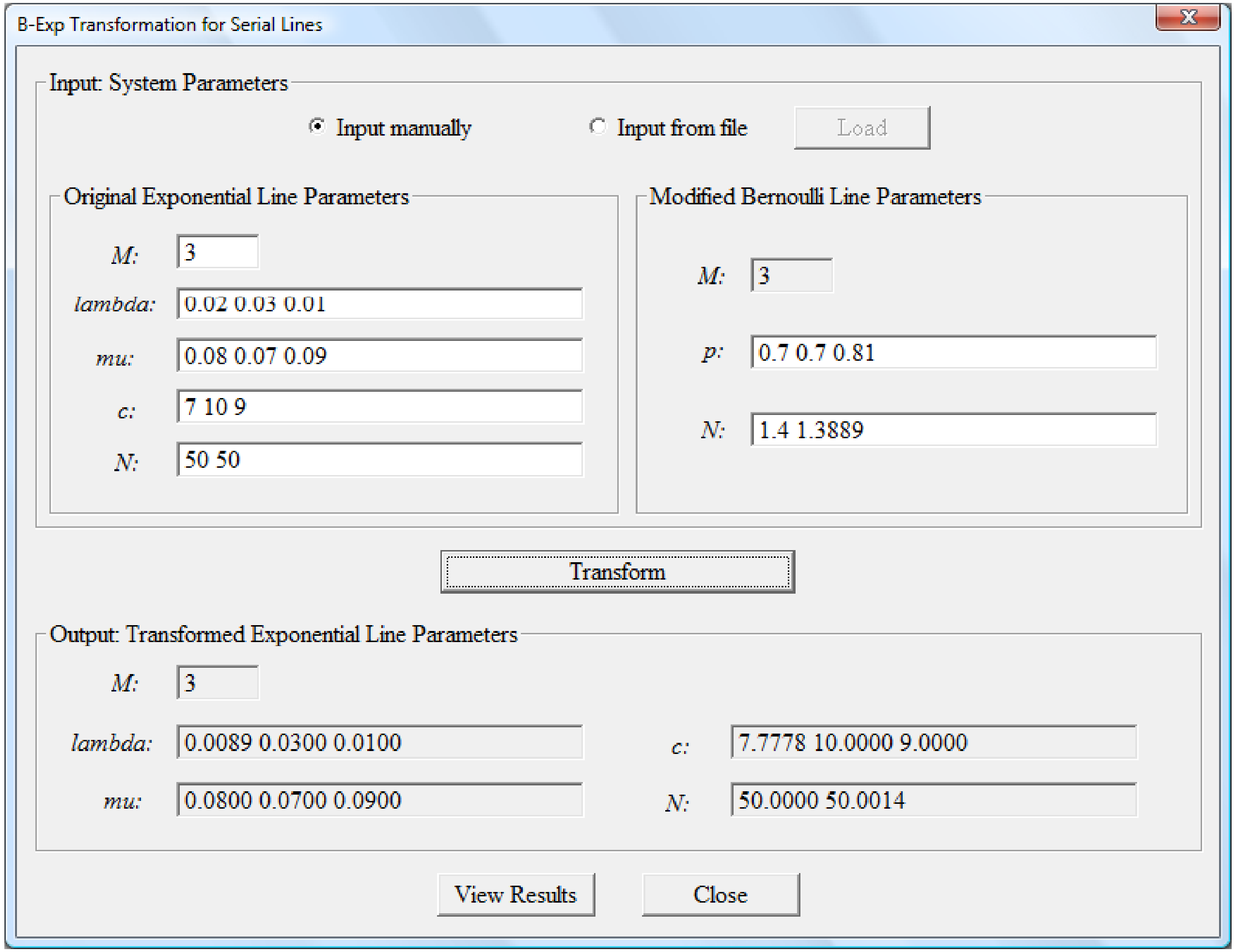
- Accuracy – the same as exp-B.

9.4 PSE Toolbox

- Exp-B transformation



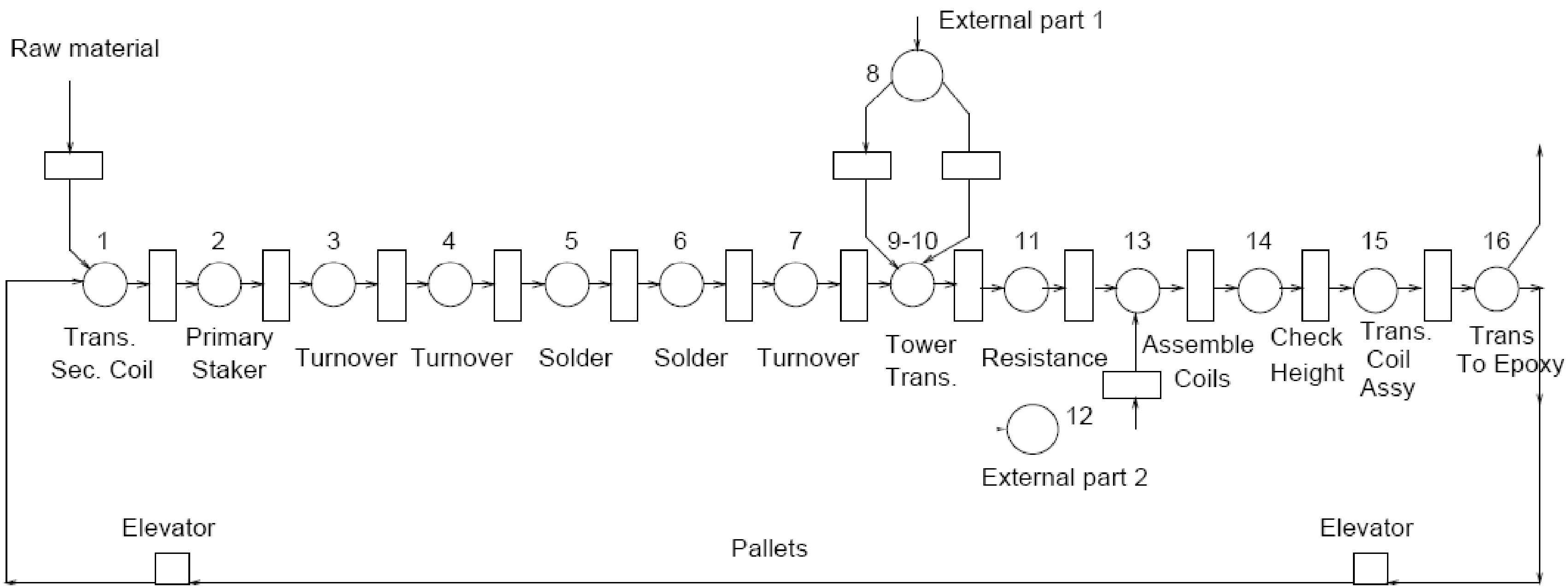
B-exp transformation



10. Case Studies

10.1 Automotive ignition coil processing system

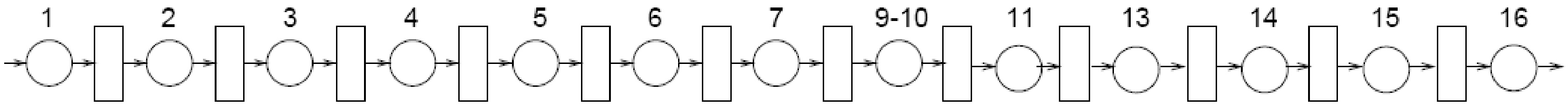
10.1.1 System layout



10.1.2 System performance

Week	Nominal throughput (parts/hr)	Actual throughput (parts/hr)	Losses (%)	Average throughput (parts/hr)	Average losses (%)
Week 1	562.53	464	17.52	472	16.10
Week 2	562.53	505	10.23		
Week 3	562.53	447	20.54		
Week 4	593.07	501	15.52	472.6	20.32
Week 5	593.07	454	23.45		
Week 6	593.07	424	28.51		
Week 7	593.07	480	19.07		
Week 8	593.07	504	15.02		

10.1.3 Structural modeling



10.1.4 Modeling and identification of the machines

	Op. 1	Op. 2	Op. 3	Op. 5	Op. 6	Op. 7
Uptime	227.79	188.11	504.15	1515.5	572.27	1493.2
Downtime	1.837	1.517	1.517	1.517	16.485	9.013

	Op. 9-10	Op. 11	Op. 14	Op. 15	Op. 16
Uptime	13.98	43.07	74.33	188.11	356.02
Downtime	1.571	1.748	1.517	1.517	2.149

(a). Period 1

	Op. 1	Op. 2	Op. 3	Op. 5	Op. 6	Op. 7
Uptime	141.18	280.31	651.73	438.67	450.37	1974
Downtime	2.15	1.976	3.275	4.879	3.632	1.976

	Op. 9-10	Op. 11	Op. 14	Op. 15	Op. 16
Uptime	16.25	45.11	52.91	168.89	201
Downtime	2.05	2.076	1.976	2.398	2.854

(b). Period 2

- Exponential assumption

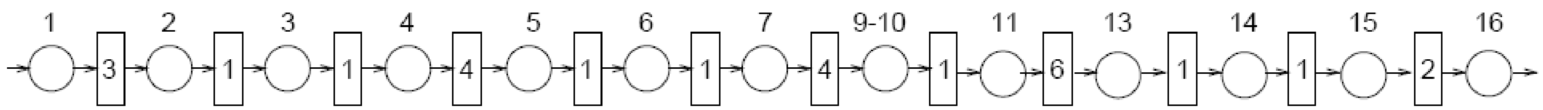
10.1.5 Modeling and identification of the buffers

Operation	Op. 1	Op. 2	Op. 3	Op. 4	Op. 5	Op. 6
Buffer capacity	3	1	1	4	1	1

Operation	Op. 7	Op. 9-10	Op. 11	Op. 13	Op. 14	Op. 15
Buffer capacity	4	1	6	1	1	2

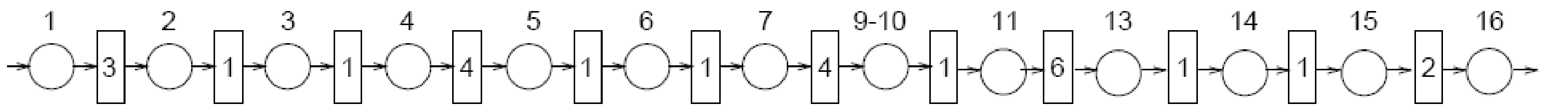
10.1.6 Overall system model

- Exponential



e_i:	0.9920	0.9920	0.9970	1.0	0.9989	0.9728	0.9937	0.8990	0.9610	1.0	0.9799	0.9920	0.9940
$T_{down;i}$	1.837	1.517	1.517	0.0	1.517	16.474	9.009	1.571	1.748	0.0	1.517	1.517	2.149

(a). Period 1

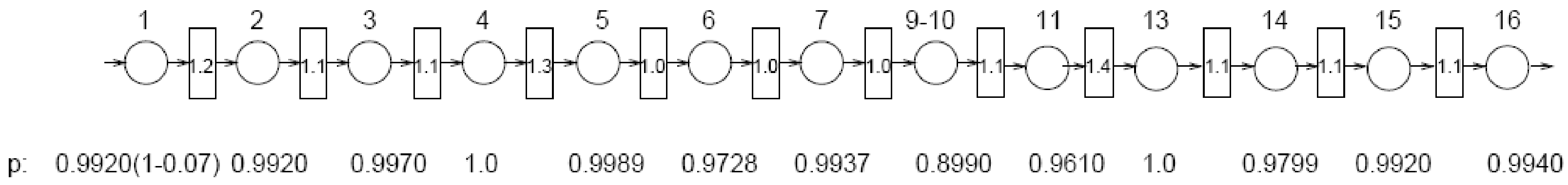


e_i:	0.985	0.993	0.995	1.0	0.989	0.992	0.999	0.888	0.956	1.0	0.964	0.986	0.986
$T_{down;i}$	2.150	1.976	3.275	0.0	4.878	3.632	1.976	2.050	2.076	0.0	1.976	2.398	2.854

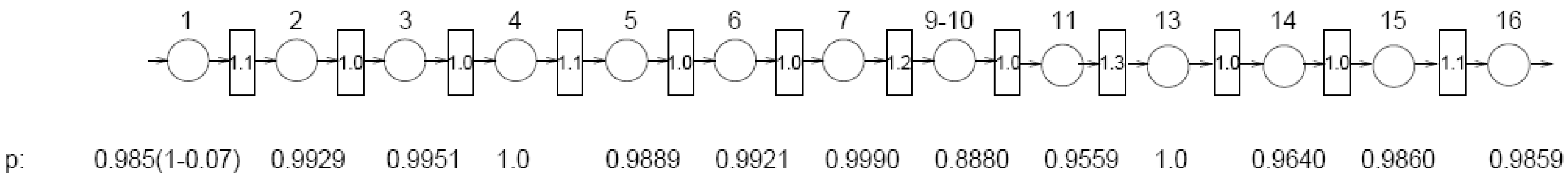
(b). Period 2

10.1.6 Overall system model (cont)

- Bernoulli



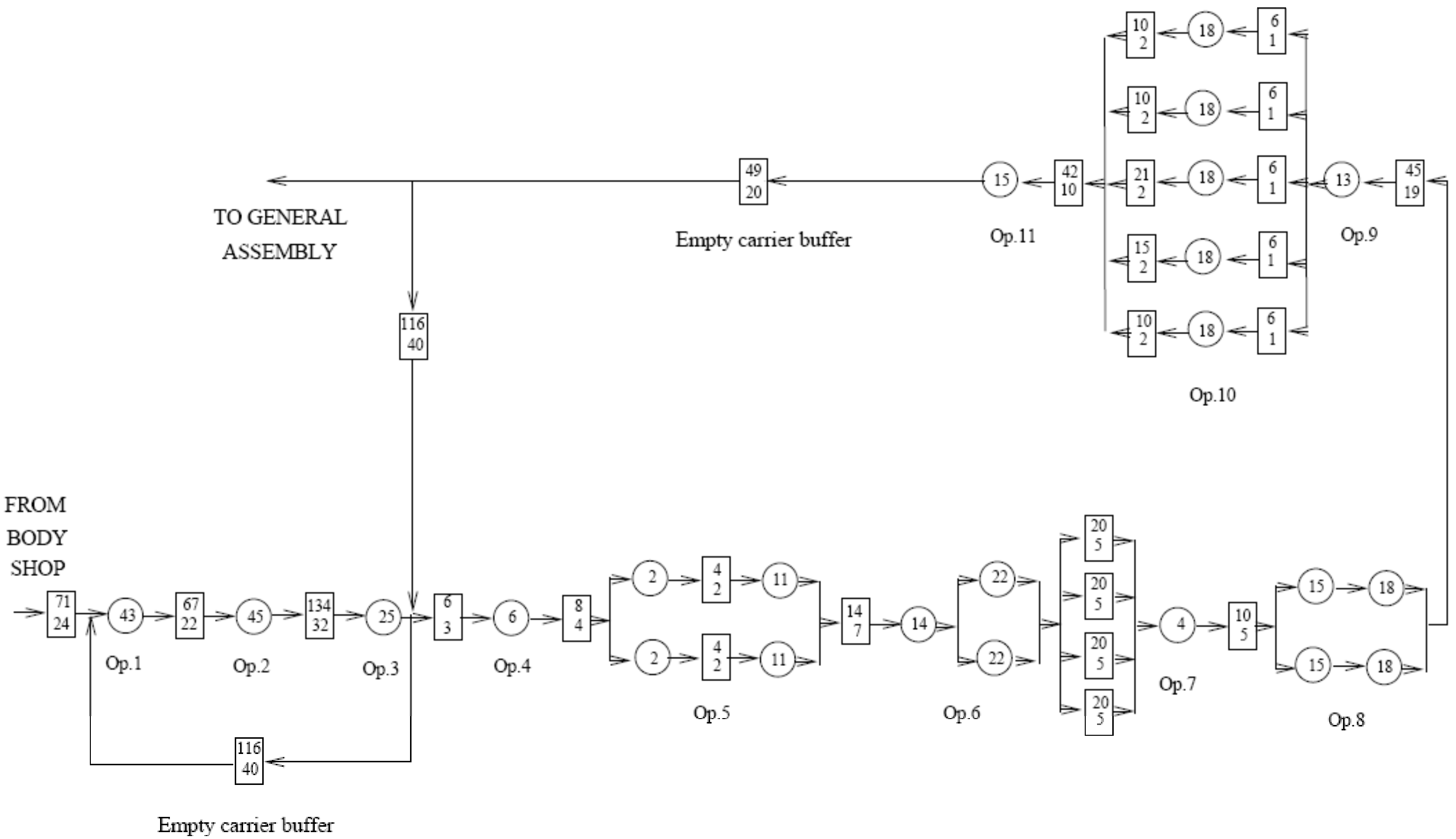
(a). Period 1



(b). Period 2

10.2 Automotive paint shop production systems

10.2.1 System layout

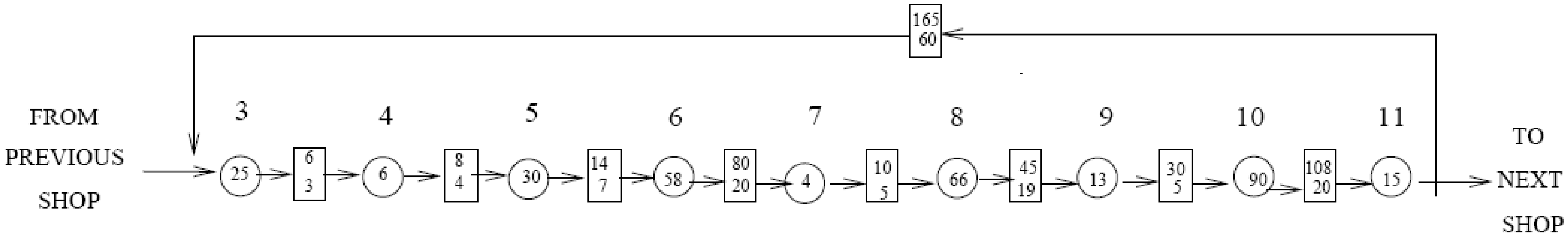


10.2.2 System performance

Ops.	1	2	3	4	5	6	7	8	9	10	11
c_i	63	63	63	63	63	63	63	63	72	100	100

Period	Month 1	Month 2	Month 3	Month 4	Month 5
TP	53.5	43.81	51.27	54.28	55.89

10.2.3 Structural modeling



10.2.4 Modeling and identification of the machines

Operation	Month 1	Month 2	Month 3	Month 4	Month 5
Op. 1	0	0.05	0.01	0	0
Op. 2	0.05	0.45	0.07	0.03	0.00
Op. 3	2.88	2.15	0.64	2.26	1.35
Op. 4	2.77	2.60	4.45	2.00	2.60
Op. 5	0.23	0.01	0.04	1.07	1.64
Op. 6	0	0	0.001	0.02	0.39
Op. 7	1.09	3.13	1.09	2.68	2.05
Op. 8	1.39	3.42	1.28	2.73	0.41
Op. 9	6.18	7.38	7.01	6.59	6.14
Op. 10	0.35	1.40	1.66	3.09	3.63
Op. 11	0.01	0.01	0.01	0.01	0.01

10.2.4 Modeling and identification of the machines (cont)

- Bernoulli assumption

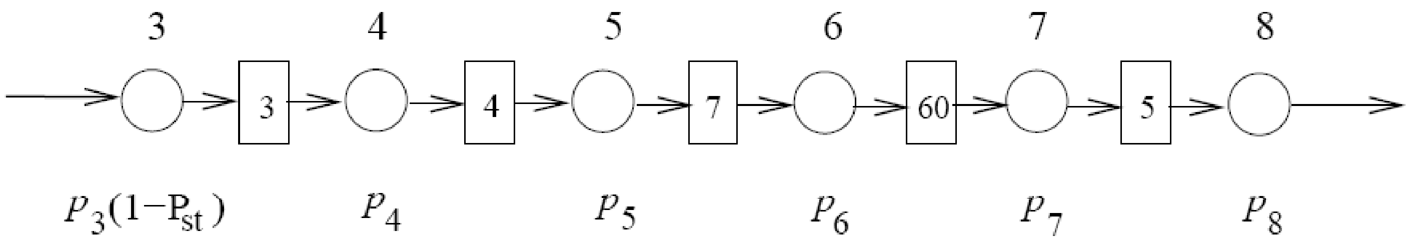
$$p_i = \min\left\{1, \frac{c_i - L_i}{63}\right\}, \qquad i = 3, \ldots, 11$$

Operations	3	4	5	6	7	8	9	10	11
Month 1	0.9543	0.9560	0.9963	1	0.9827	0.9779	1	1	1
Month 2	0.9659	0.9587	0.9998	1	0.9503	0.9457	1	1	1
Month 3	0.9898	0.9294	0.9994	1	0.9827	0.9797	1	1	1
Month 4	0.9641	0.9683	0.9830	0.9997	0.9575	0.9567	1	1	1
Month 5	0.9786	0.9587	0.9740	0.9938	0.9675	0.9935	1	1	1

10.2.5 Modeling and identification of the buffers

Operations	3	4	5	6	7	8	9	10
N_i	3	4	7	60	5	26	25	88

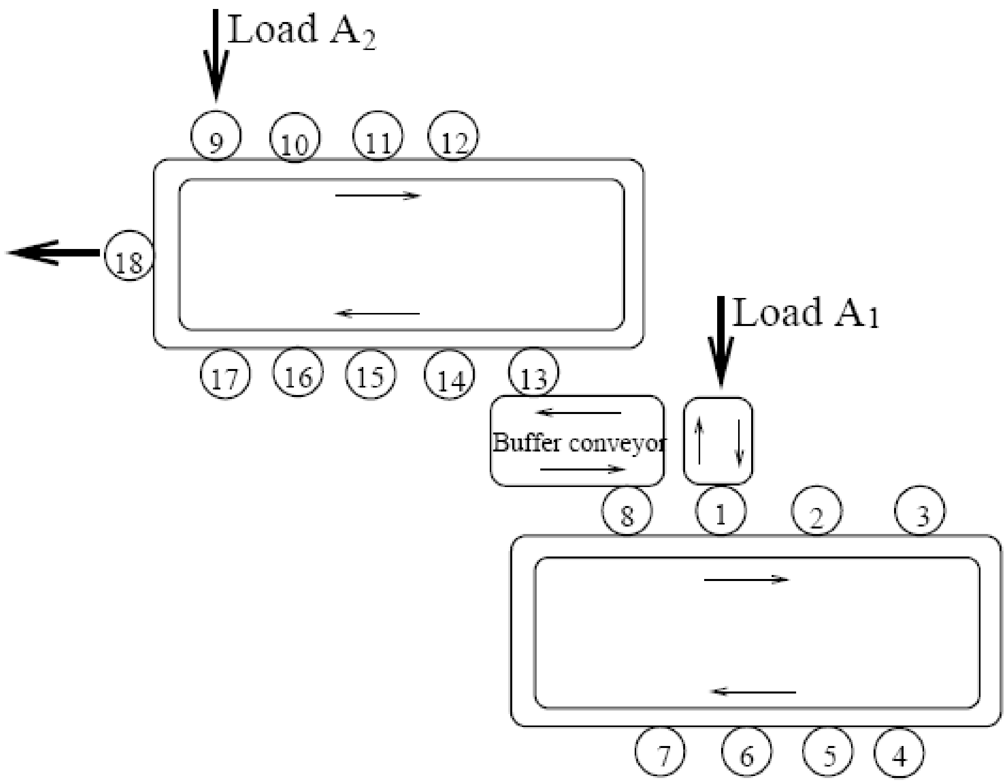
10.2.6 Overall system model



	Month 1	Month 2	Month 3	Month 4	Month 5
P_{st}	0.0981	0.1171	0.1113	0.1046	0.0975

10.3 Automotive ignition module assembly system

10.3.1 System layout

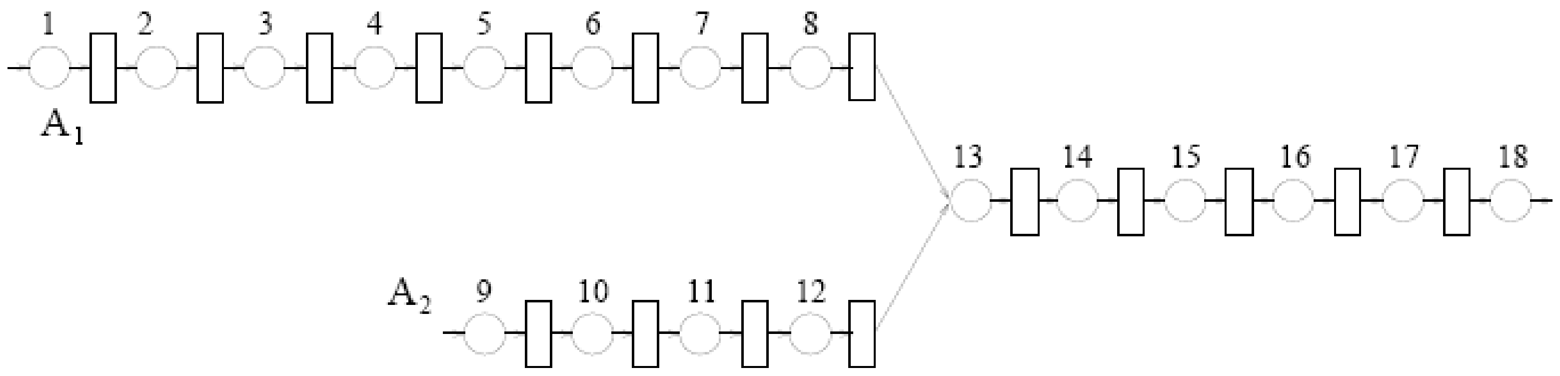


10.3.2 System performance

Month	May	June	July	Aug.	Sep.	Oct.
TP (parts/hr)	337	347	378	340	384	383

- Numerical throughput 600 part/hour.
- Losses ≈ 40%.

10.3.3 Structural modeling



10.3.3 Structural modeling (cont.)

- Exponential assumption
- $\tau_i \approx 6$ sec/part

Operations	May		June		July	
	T_{down} (min)	T_{up} (min)	T_{down} (min)	T_{up} (min)	T_{down} (min)	T_{up} (min)
1	4.8	38.8	8.4	159.6	18.3	286.7
2	3.0	35.4	1.7	55	3.7	42.6
3	4.5	70.5	5.7	136.8	12.4	142.6
4	10.2	68.3	7.1	57.4	12.7	66.7
5	8.9	65.3	6.3	56.7	12.7	66.7
6	2.0	98	6.7	216.6	5.2	59.8
7	1.8	58.2	4.4	142.3	12.2	231.8
8	2.5	47.5	2.4	37.6	5.7	65.6
9	3.9	31.6	7.1	81.7	7.3	65.7
10	2.6	34.5	3.3	33.4	4.0	53.1
11	2.7	31.1	3.4	45.2	4.1	54.5
12	3.3	326.7	0.9	89.1	16.9	224.5
13	3.8	91.2	1.6	38.4	17.9	144.8
14	5.2	98.8	2.5	47.5	10.2	103.1
15	1.8	14.6	2.8	90.5	16.8	223.2
16	2.8	137.2	10.8	529.2	23.4	269.1
17	1.7	55	10.3	504.7	27.7	368
18	2.2	107.8	1.8	20.7	3.4	30.6

10.3.4 Modeling and identification of the buffers

Operations	1	2	3	4	5	6	7	8	9
Buffer capacity	5	5	7	5	5	5	8	8	5

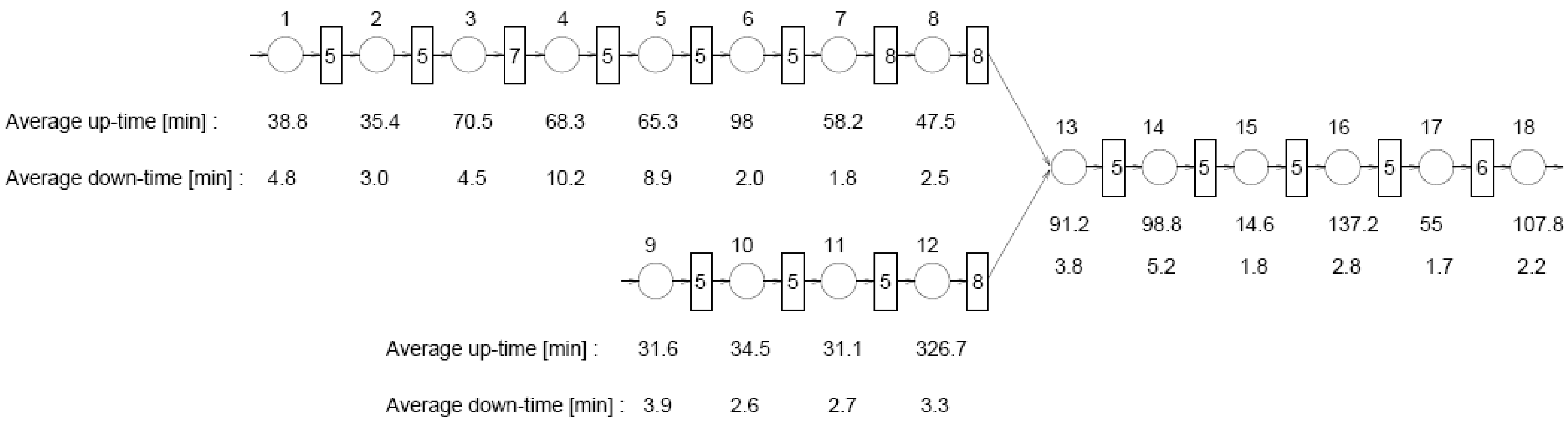
Operations	9	10	11	12	13	14	15 16	17	18
Buffer capacity	5	5	8	5	5	5	5	6	

10.3.5 Average frequency of starvations and blockages by carriers

Month	May	June	July	Aug.	Sep.	Oct.
Average fraction of time when Op. 1 is starved	0.257	0.308	0.285	0.28	0.199	0.252
Average fraction of time when Op. 9 is starved	0.134	0.247	0.199	0.099	0.142	0.089
Average fraction of time when Op. 18 is blocked	0.236	0.256	0.226	0.189	0.11	0.238

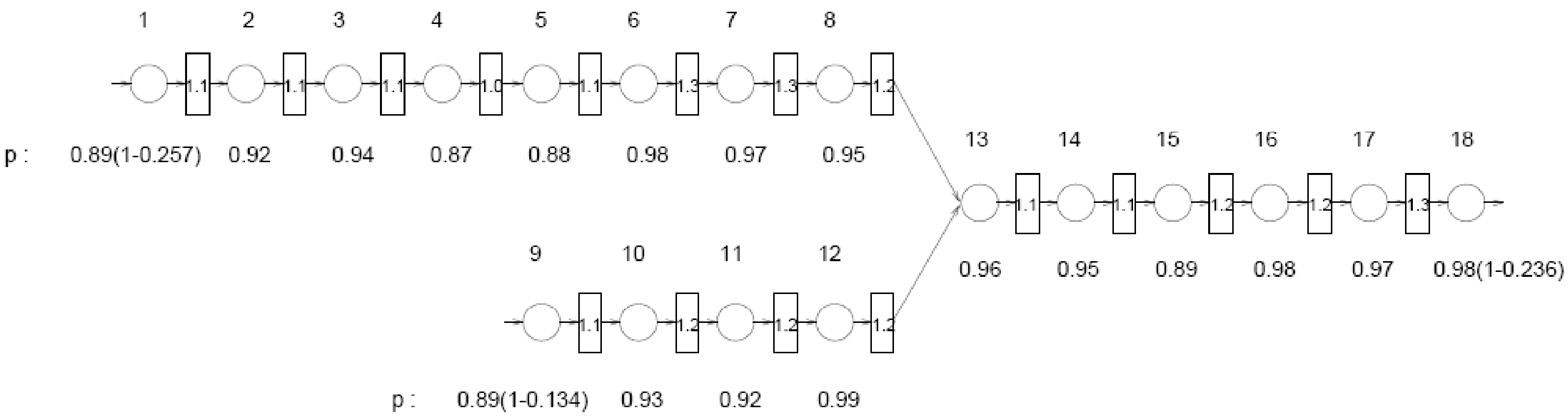
10.3.6 Overall system model

- Exponential



10.3.6 Overall system model (cont)

- Bernoulli



11. Summary

11.1 Steps of mathematical modeling

- Layout
- Structural modeling
- Machine identification
- Buffer identification
- Model validation

11.2 Machine model

- Cycle time
- pmf or pdf of uptime
- pmf or pdf of downtime
- pmf or pdf of parts quality

11.3 Reliability models

- Bernoulli
- Geometric
- Exponential
- Rayleigh
- Weibull
- Gamma
- Log-normal
- General

11.4 Modeling conventions

- Slotted or unslotted time
- Time-dependent or operation-dependent
- Synchronous or asynchronous
- Discrete event or flow model
- Blocked before or blocked after service

11.5 Performance measures

- Production rate or throughput
- Work-in-process
- Finished goods inventory
- Probability of blockages and starvations
- Customer demand satisfaction
- Transient properties