Chapter 4

EVALUATION OF THROUGHPUT IN SERIAL PRODUCTION LINES WITH NON-EXPONENTIAL MACHINES

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Abstract This paper provides an analytical method for evaluating production rates in serial lines having finite buffers and unreliable machines with arbitrary unimodal distributions of up- and downtime. Provided that each buffer is capable of accommodating at least one downtime of all machines in the system, we show that the production rate (a) is relatively insensitive to the type of up- and downtime distributions and (b) can be approximated by a linear function of their coefficients of variation. The results obtained are verified using Weibull, gamma, and log-normal probability distributions of up- and downtime.

1. Introduction

Analytical methods for evaluating throughput in serial production lines are available only if the up- and downtime of machines obey either exponential (in discrete time, geometric) or coaxial (phase type) probability distributions (see reviews by Koenigsberg, 1959; Buxey et al., 1973; Buzacott and Hanifin, 1978; Dallery and Gershwin, 1992; Papadopoulos and Heavey, 1996, monographs by Viswanadham and Narahari (1992); Buzacott and Shanthikumar (1993); Gershwin (1994); Altiok (1997) and representative papers by Sevast'yanov (1962); Buzacott (1967); Sheskin (1976); Soytser et al. (1979); Wijingaard (1979); Gershwin and Berman (1981); Altiok (1985, 1989); Buzacott and Kotelski (1987); Choong and Gershwin (1987); Gershwin (1987); Jafari and Shanthikumar (1987); De Koster (1987, 1988); Terracol and David (1987); Dallery et al. (1988, 1989); Altiok and Ranjan (1989); Liu and Buzacott (1989); Lim et al. (1990); Hiller and So (1991a,b); Glassey and Hong (1993); Powell (1994);

Jacobs and Meerkov (1995a,b); Tan and Yeralan (1997); Chiang et al. (1998, 2000, 2001); Yamshita and Altiok (1998); Dallery and Le Bihan (1999); Vidalis and Papdopoulos (1999); Tempelmeier and Burger (2001); Enginarlar et al. (2002); Sadr and Malhame (2003); Tempelmeier (2003)). The present paper is intended to offer an analytical method for calculating throughput in serial lines with machines having arbitrary unimodal distributions of up- and downtime, provided that each buffer is capable of accommodating at least one downtime of all machines in the system. Specifically, we show that the production rate, PR, of a serial line (i.e., the average number of parts produced by the last machine per unit of time) can be evaluated as follows:

$$\begin{gathered} \mathrm{PR} = e_{\min} - (e_{\min} - \mathrm{PR}^{\exp}) \sum_{i=1}^{M} \frac{\mathrm{CV}_{\mathrm{up},i} + \mathrm{CV}_{\mathrm{down},i}}{2M}, \\ \mathrm{CV}_{\mathrm{up},i} \in [0,1], \quad \mathrm{CV}_{\mathrm{down},i} \in [0,1], \end{gathered} \tag{4.1}$$

where $\mathrm{PR}^{\exp}$ is the production rate of the line if all machines were exponential, $\mathrm{CV}_{\mathrm{up},i}$ and $\mathrm{CV}_{\mathrm{down},i}$ are the coefficients of variation of up- and downtime of the ith machine, $i = 1, \dots, M$, and $e_{\min}$ is the smallest efficiency in isolation among all the machines in the system, i.e.,

$e_{\min} = \min_{i=1,\dots,M} e_i = \min_{i=1,\dots,M} \frac{T_{\mathrm{up},i}}{T_{\mathrm{up},i}+T_{\mathrm{down},i}}$,

$T_{\mathrm{up},i} =$ average uptime of machine i,

$T_{\mathrm{down},i} =$ average downtime of machine i,

$\mathrm{CV}_{\mathrm{up},i} = \frac{\sigma_{\mathrm{up},i}}{T_{\mathrm{up},i}}$, $\mathrm{CV}_{\mathrm{down},i} = \frac{\sigma_{\mathrm{down},i}}{T_{\mathrm{down},i}}$,

$\sigma_{\mathrm{up},i} =$ standard deviation of uptime of machine i,

$\sigma_{\mathrm{down},i} =$ standard deviation of downtime of machine i,

$M =$ number of machines in the system.

Using Weibull, gamma, and log-normal probability distributions, we show that the accuracy of this method is within 6%. Along with providing a quantitative result, expression (4.1) indicates that the production rate depends mostly on the first two moments of up- and downtime, rather than on complete distributions of these random variables.

The CVs considered in this paper are less than 1 because, according to the empirical evidence of Inman (1999), the equipment on the factory floor often satisfies this condition. In addition, it has been shown in Li and Meerkov (2003) that CVs are less than 1 if the breakdown and

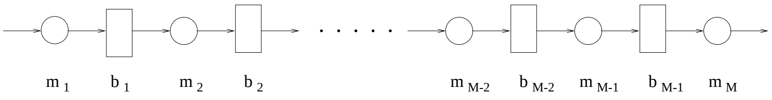


Figure 4.1. Serial production line.

repair rates of the machines are increasing functions of time, which often take place in reality.

The outline of this paper is as follows: In Section 2, the model of the production system under consideration is introduced and the problems addressed are formulated. Sections 3 and 4 introduce some analytical expressions and describe the approach of the study, respectively. Sections 5–7 present the main results, and in Section 8 the conclusions are formulated. The proofs are given in the Appendix.

2. Model and problem formulation

2.1 Model

The block diagram of the production system considered in this work is shown in Figure 4.1, where the circles represent the machines and the rectangles are the buffers. Assumptions on the machines and buffers are as follows:

(i) Each machine m_i, $i = 1, \ldots, M$, has two states: up and down. When up, the machine is capable of processing one part per cycle time; when down, no production takes place. The cycle times of all machines are the same.

(ii) The up- and downtime of each machine are continuous random variables, $t_{\text{up},i}$ and $t_{\text{down},i}$, $i = 1, \ldots, M$, with arbitrary unimodal probability density functions, $f_{t_{\text{up},i}}(t)$ and $f_{t_{\text{down},i}}(t)$, $t \geq 0$, $i = 1, \ldots, M$, respectively. It is assumed that these random variables are mutually independent. For convenience, it is also assumed that the up- and downtime are measured in units of the cycle time. In other words, uptime (respectively, downtime) of length $t \geq 0$ implies that the machine is up (respectively, down) during t cycle times.

(iii) Buffer b_i, $i = 1, \ldots, M-1$, is of capacity N_i such that $\max_{i=1,\ldots,M} T_{\text{down},i} \leq N_i < \infty$, where $T_{\text{down},i}$ is the average downtime of machine m_i. In other words, it is assumed that

$$k_i = \frac{N_i}{\max_{i=1,\ldots,M} T_{\text{down},i}} \geq 1, \tag{4.2}$$

where parameter k_i is referred to as the level of buffering.

(iv) Machine m_i, $i = 2, \ldots, M$, is starved at time t if it is up at time t, buffer b_{i-1} is empty at time t and m_{i-1} does not place any work in this buffer at time t. Machine m_1 is never starved.

(v) Machine m_i, $i = 1, \ldots, M-1$, is blocked at time t if it is up at time t, buffer b_i is full at time t and m_{i+1} fails to take any work from this buffer at time t. Machine m_M is never blocked.

The production rate, PR, of the serial line (i)–(v) is the average number of parts produced by the last machine, m_M, per cycle time. As it was pointed out above, no analytical method for its evaluation are available in the literature, except for exponential and coaxial distributions of up- and downtime of the machines.

2.2 Notations

Each machine considered in this paper is denoted by a pair

$$[f_{t_{\text{up},i}}, f_{t_{\text{down},i}}], \quad i = 1, \ldots, M, \tag{4.3}$$

where, as before, $f_{t_{\text{up},i}}$ and $f_{t_{\text{down},i}}$ are the probability density functions of up- and downtime of machine i, respectively. The serial line with M machines is denoted as

$$\{[f_{t_{\text{up},1}}, f_{t_{\text{down},1}}], \ldots, [f_{t_{\text{up},M}}, f_{t_{\text{down},M}}]\}. \tag{4.4}$$

If all machines have identical distributions of up- and downtime, the notation for the line is:

$$\{[f_{t_{\text{up}}}, f_{t_{\text{down}}}]_i, i = 1, \ldots, M\}. \tag{4.5}$$

2.3 Problems addressed

Using the model (i)–(v) and notations (4.4), (4.5), this paper is intended to:

- Develop an analytical method for calculating the production rate in serial lines (4.4) and (4.5) under the assumption that the average uptime and downtime of all machines are identical and, in addition, the coefficients of variation of uptime and downtime of all machines are the same and, moreover, equal to each other, i.e.,

$$\begin{gathered} T_{\text{up},i} = T_{\text{up}}, \quad T_{\text{down},i} = T_{\text{down}}, \qquad i = 1, \ldots, M, \\ \text{CV}_{\text{up},i} = \text{CV}_{\text{down},i} = \text{CV}, \quad i = 1, \ldots, M. \end{gathered} \tag{4.6}$$

This is referred to as the case of *identical machines.* Note that the machines may have different distributions of up- and downtime but are identical in the sense (2.3).

- Extend this method to the case where $T_{\mathrm{up},i}$ and $T_{\mathrm{down},i}$ are arbitrary and the coefficients of variation of uptime and downtime of all machines are the same but may be nonequal to each other, i.e.,

  $$\mathrm{CV}_{\mathrm{up},i} = \mathrm{CV}_{\mathrm{up}}, \quad \mathrm{CV}_{\mathrm{down},i} = \mathrm{CV}_{\mathrm{down}}, \qquad i = 1, \ldots, M, \quad (4.7)$$

  and, in general,

  $$\mathrm{CV}_{\mathrm{up}} \neq \mathrm{CV}_{\mathrm{down}}.$$

  This is referred to as the case of *identical coefficients of variation.*

- Finally, extend this method to the case where all $T_{\mathrm{up},i}$, $T_{\mathrm{down},i}$, $\mathrm{CV}_{\mathrm{up},i}$ and $\mathrm{CV}_{\mathrm{down},i}$, $i = 1, \ldots, M$, are arbitrary. This is referred to as the *general case.*

3. Analytical expressions

3.1 Production rate for $\mathrm{CV}_{\mathrm{up}} = \mathrm{CV}_{\mathrm{down}} = 0$

In the case of $\mathrm{CV}_{\mathrm{up}} = \mathrm{CV}_{\mathrm{down}} = 0$, the production rate of the line (i)–(v) can be evaluated as follows:

THEOREM 4.1 *Consider a serial production line defined by assumption* (i)–(v) *and assume that* $\mathrm{CV}_{\mathrm{up}} = \mathrm{CV}_{\mathrm{down}} = 0$. *Then its production rate is given by*

$$\mathrm{PR} = \min_{i=1,\ldots,M} \frac{T_{\mathrm{up},i}}{T_{\mathrm{up},i} + T_{\mathrm{down},i}},$$

i.e., the PR *of the line is equal to the smallest efficiency in isolation among all machines in the system.*

Proof. See the Appendix. □

It should be pointed out that the main reason why Theorem 4.1 holds is that the level of buffering $k_i \geq 1$, $\forall i = 1, \ldots, M-1$.

3.2 Production rate for $\mathrm{CV}_{\mathrm{up}} = \mathrm{CV}_{\mathrm{down}} = 1$

Assume that all machines have up- and downtime distributed exponentially and, therefore, $\mathrm{CV}_{\mathrm{up}} = \mathrm{CV}_{\mathrm{down}} = 1$. As it was pointed above, PR in serial line (i)–(v) with exponential machines can be evaluated using a number of analytical techniques instance, (see, for Gershwin, 1987; De Koster, 1987, 1988; Dallery et al., 1988, 1989; Chiang et al., 2000, 2001; Sadr and Malhame, 2003).

Although all of them are relatively precise, each has a certain error in comparison with the real production rate (which can be obtained, for example, by numerical simulations). Because of this error, to determine the accuracy of the method developed in this paper, we evaluate PR of serial lines with exponential machines using simulations, rather than analytical calculations. We denote this production rate as $\mathrm{PR}^{\mathrm{exp}}$.

3.3 Production rate for $0 < \mathrm{CV}_{\mathrm{up}}, \mathrm{CV}_{\mathrm{down}} < 1$

Based on the above, PR of serial lines with CV = 0 and CV = 1 can be easily evaluated. For all other CVs, we formulate

Hypothesis 4.1 *In the case of identical machines* (2.3)*, the production rate of serial lines* (i)–(v) *can be evaluated as follows*:

$$\mathrm{PR} = e - (e - \mathrm{PR}^{\mathrm{exp}})\mathrm{CV}, \tag{4.8}$$

where e is the machine efficiency in isolation, i.e., $e = \frac{T_{\mathrm{up}}}{T_{\mathrm{up}}+T_{\mathrm{down}}}$.

Hypothesis 4.2 *In the case of identical coefficients of variation* (4.7)*, the production rate of serial lines* (i)–(v) *can be evaluated as follows*:

$$\mathrm{PR} = e_{\min} - (e_{\min} - \mathrm{PR}^{\mathrm{exp}})\frac{\mathrm{CV}_{\mathrm{up}} + \mathrm{CV}_{\mathrm{down}}}{2}, \tag{4.9}$$

where $e_{\min} = \min_{i=1,\dots,M} e_i = \min_{i=1,\dots,M} \frac{T_{\mathrm{up},i}}{T_{\mathrm{up},i}+T_{\mathrm{down},i}}$.

Hypothesis 4.3 *In the general case, the production rate of serial lines* (i)–(v) *can be evaluated as follows*:

$$\mathrm{PR} = e_{\min} - (e_{\min} - \mathrm{PR}^{\mathrm{exp}})\sum_{i=1}^{M}\frac{\mathrm{CV}_{\mathrm{up},i} + \mathrm{CV}_{\mathrm{down},i}}{2M}. \tag{4.10}$$

Verifications of these Hypotheses are given in Sections 5–7, while the approach to the verification is described in Section 4.

4. Approach

4.1 Distributions considered

For the verification of (4.8)–(4.10) we consider the following distributions:

(a) Weibull, i.e.,

$$\begin{aligned} f_{t_{\mathrm{up},i}}(t) &= p^P e^{-(pt)^P} P t^{P-1}, \\ f_{t_{\mathrm{down},i}}(t) &= r^R e^{-(rt)^R} R t^{R-1}. \end{aligned} \tag{4.11}$$

Here and in all subsequent distributions, (p, P) and (r, R) are positive real numbers. These distributions are denoted as $W(p, P)$ and $W(r, R)$, respectively.

(b) Gamma, i.e.,

$$f_{t_{\text{up},i}}(t) = pe^{-pt}\frac{(pt)^{P-1}}{\Gamma(P)}, \quad f_{t_{\text{down},i}}(t) = re^{-rt}\frac{(rt)^{R-1}}{\Gamma(R)}, \tag{4.12}$$

where $\Gamma(x)$ is the gamma function, $\Gamma(x) = \int_0^\infty s^{x-1}e^{-s}ds$. These distributions are denoted as $g(p, P)$ and $g(r, R)$, respectively.

(c) Log-normal, i.e.,

$$\begin{aligned} f_{t_{\text{up},i}}(t) &= \frac{1}{\sqrt{2\pi}Pt}e^{-\frac{(ln(t)-p)^2}{2P^2}}, \\ f_{t_{\text{down},i}}(t) &= \frac{1}{\sqrt{2\pi}Rt}e^{-\frac{(ln(t)-r)^2}{2R^2}}. \end{aligned} \tag{4.13}$$

We denote these distributions as $\text{LN}(p, P)$ and $\text{LN}(r, R)$, respectively.

Specific realizations of downtime distributions analyzed in this work are given in Table 4.1. They are classified according to their coefficients of variation, CV_{down}, which take values from the set $\{0.1,0.25,0.5,0.75,1\}$, and according to their average values, which are 10 and 20.

Table 4.1. Downtime distributions considered

CV_{down}	$T_{\text{down}} = 10$
0.1	$W(0.0959, 12.15)$, $g(10, 100)$, $\text{LN}(2.30, 0.1)$
0.25	$W(0.0913, 4.542)$, $g(1.6, 16)$, $\text{LN}(2.27, 0.25)$
0.5	$W(0.0886, 2.1013)$, $g(0.4, 4)$, $\text{LN}(2.19, 0.47)$
0.75	$W(0.0917, 1.3475)$, $g(0.18, 1.78)$, $\text{LN}(2.08, 0.67)$
1.00	$\text{LN}(1.96, 0.83)$

CV_{down}	$T_{\text{down}} = 20$
0.1	$W(0.0479, 12.15)$, $g(5, 100)$, $\text{LN}(2.99, 0.1)$
0.25	$W(0.0457, 4.542)$, $g(0.8, 16)$, $\text{LN}(2.97, 0.25)$
0.5	$W(0.0443, 2.1013)$, $g(0.2, 4)$, $\text{LN}(2.88, 0.47)$
0.75	$W(0.0459, 1.3475)$, $g(0.09, 1.78)$, $\text{LN}(2.77, 0.67)$
1.00	$\text{LN}(2.65, 0.83)$

The uptime distributions, corresponding to the downtime distributions of Table 4.1, have been selected as follows:

For the case of identical machines, given machine efficiency, e, the average uptime was chosen as

$$T_{\text{up}} = \frac{e}{1-e}T_{\text{down}}.$$

Next, $\mathrm{CV}_{\mathrm{up}}$ was selected as $\mathrm{CV}_{\mathrm{up}} = \mathrm{CV}_{\mathrm{down}}$ and, using these T_{up} and $\mathrm{CV}_{\mathrm{up}}$, the distribution of uptime was selected to be the same as that of the downtime, if the case of identical distributions was analyzed; otherwise it was selected randomly and equiprobably from the set $\{W, g, \mathrm{LN}\}$.

For the case of non-identical machines (4.7), the values of e_i, $T_{\mathrm{down},i}$ and the distributions of up- and downtime were selected randomly and equiprobably from the sets {0.55, 0.65, 0.75, 0.85, 0.9, 0.95}, $\{10, 20\}$ and $\{w, g, \mathrm{LN}\}$, respectively.

4.2 Evaluation of the production rate

To evaluate the production rate of serial lines (i)–(v) with up- and downtime distributed according to the distributions described above, a `MATLAB` code was constructed, which simulated the operation of the production line (i)–(v). In all simulation runs, zero initial conditions of all buffers have been assumed and the states of all machines at the initial time moment have been selected "up". The first 10,000 cycle times were considered the warm-up period. The subsequent 100,000 cycle times were used for statistical evaluation of PR. Each simulation was repeated 20 times, which resulted in 95% confidence intervals of less than 0.003.

4.3 Parameters selected

In all systems analyzed, particular values of M, e, and N have been selected as follows:

(a) The number of machines in the system, M: The number of machines in the system was selected to be 3, 5 and 10.

(b) Machine efficiency, e: Although in practice e may have widely different values (e.g., smaller in machining operations and much larger in assembly), to obtain a manageable set of systems, e was selected from the set $\{0.55, 0.65, 0.75, 0.85, 0.9, 0.95\}$.

(c) Level of buffering, k_i: the value of k_i was selected to be 1 ("small" buffer capacity) or 3 ("large" buffer capacity).

4.4 Systems analyzed

We consider two groups of systems. The first one consists of machines with identical types of up- and downtime distributions. For the case of identical machines, this group is given by

$$\begin{gathered}\{[W(p,P), W(r,R)]_i, i = 1, \dots, 10\},\\ \{[g(p,P), g(r,R)]_i, i = 1, \dots, 10\},\\ \{[\mathrm{LN}(p,P), \mathrm{LN}(r,R)]_i, i = 1, \dots, 10\}.\end{gathered} \tag{4.14}$$

We use systems (4.14) in order to evaluate the sensitivity of PR to different distributions of up- and downtime.

For the case of identical coefficients of variation and for the general case, this group is denoted as

$$\begin{gathered}\{[W(p_1,P_1),W(r_1,R_1)],\dots,[W(p_{10},P_{10}),W(r_{10},R_{10})]\},\\ \{[g(p_1,P_1),g(r_1,R_1)],\dots,[g(p_{10},P_{10}),g(r_{10},R_{10})]\},\\ \{[\mathrm{LN}(p_1,P_1),\mathrm{LN}(r_1,R_1)],\dots,[\mathrm{LN}(p_{10},P_{10}),\mathrm{LN}(r_{10},R_{10})]\}.\end{gathered}\tag{4.15}$$

The second group consists of machines with different up- and downtime distributions. These lines have been formed as follows: For each machine m_i, $i = 1,\dots,M$, the up- and downtime distributions were chosen from the set $\{W, g, \mathrm{LN}\}$ equiprobably and independently of each other and all other machines in the system. As a result, the following lines have been selected:

$$\begin{aligned}\text{Line 1: }&\{[g,W],[\mathrm{LN},\mathrm{LN}],[W,g],[g,\mathrm{LN}],[g,W],[\mathrm{LN},g],\\ &\qquad[W,W],[g,g],[\mathrm{LN},W],[g,\mathrm{LN}]\}\\ \text{Line 2: }&\{[W,\mathrm{LN}],[g,W],[\mathrm{LN},W],[W,g],[g,\mathrm{LN}],[g,W],\\ &\qquad[W,W],[\mathrm{LN},g],[g,W],[\mathrm{LN},\mathrm{LN}]\}\end{aligned}\tag{4.16}$$

For $M = 3$ (respectively, $M = 5$), the first 3 (respectively, first 5) machines of lines (4.14)–(4.16) have been used.

We will employ the notations $A \in \{(4.14)\}$ or $A \in \{(4.14), (4.16)\}$ or $A \in \{(4.15), (4.16)\}$ to indicate, respectively, that line A is one of (4.14) or one of (4.14), (4.16) or one of (4.15), (4.16).

Specific parameters of the distributions involved in (4.14)–(4.16) are selected in a manner consistent with the problem analyzed; they are described in Subsections 5.1, 6.1 and 7.1.

4.5 Metrics for sensitivity and accuracy analysis

The analysis of the sensitivity of PR to the type of up- and downtime distribution is carried out using the following metric:

$$\varepsilon = \max_{A,B\in\{(4.14)\}} \frac{|\mathrm{PR}^A - \mathrm{PR}^B|}{\mathrm{PR}^A}\cdot 100\%,\tag{4.17}$$

where PR^A and PR^B are the production rates of systems from set (4.14) evaluated by simulations.

The accuracy of Hypothesis i, $i = 1, 2, 3$, is estimated using the following metrics:

$$\Delta_1 = \max_{A\in\{(4.14),(4.16)\}} \frac{|\mathrm{PR}^A - \mathrm{PR}_1|}{\mathrm{PR}^A}\cdot 100\%,\tag{4.18}$$

$$\Delta_i = \max_{A \in \{(4.15),(4.16)\}} \frac{|\mathrm{PR}^A - \mathrm{PR}_i|}{\mathrm{PR}^A} \cdot 100\%, \quad i = 2, 3, \tag{4.19}$$

where PR^A is, as before, the production rates of line from (4.14), (4.16) or (4.15), (4.16) evaluated by simulation and PR_i, $i = 1, 2, 3$, is the production rate calculated using Hypothesis i.

5. Production rate evaluation for the case of identical machines

5.1 Parameters of systems analyzed

Since in this case all machines have identical T_{up}, T_{down}, $\mathrm{CV}_{\mathrm{up}}$ and $\mathrm{CV}_{\mathrm{down}}$ and, moreover, $\mathrm{CV}_{\mathrm{up}} = \mathrm{CV}_{\mathrm{down}} = \mathrm{CV}$, the parameters of the systems analyzed coincide with those introduced in Section 4, i.e.,

$$\begin{aligned} \mathrm{CV} &\in \{0.1, 0.25, 0.5, 0.75, 1\}, \\ T_{\mathrm{down}} &\in \{10, 20\}, \\ e &\in \{0.55, 0.65, 0.75, 0.85, 0.9, 0.95\}, \\ M &\in \{3, 5, 10\}, \\ k &\in \{1, 3\}. \end{aligned}$$

Taking into account that these parameters have been used for all five systems (4.14), (4.16), this implies that the total of 1800 different production lines have been analyzed.

5.2 Results

Tables 4.2 and 4.3 present the production rates of serial lines (4.14) and (4.16), evaluated by simulations and by Hypothesis 4.1 (broken lines). In these Tables, the rows and columns correspond to $e \in \{0.55, 0.65, 0.75, 0.85, 0.9, 0.95\}$ and $M \in \{3, 5, 10\}$, respectively. Each entry of the Tables contains the data for $k = 1$ and $k = 3$. Based on these data, we conclude the following:

(a) The type of up- and downtime distributions does not affect PR in any significant manner. This phenomenon is quantified by the values of metric ε (calculated according to (4.17)) given in Tables 4.4 and 4.5. As one can see, they take values within 6%. In addition, Tables 4.4 and 4.5 exhibits qualitative effects of system parameters on the sensitivity of PR to distributions of up- and downtime. These effects can be summarized as follows:

$$\begin{aligned} \mathrm{CV} \uparrow &\Rightarrow \text{ Sensitivity } \uparrow, \\ M \uparrow &\Rightarrow \text{ Sensitivity } \uparrow, \end{aligned}$$

Table 4.2. Production rates evaluated by simulations and by Hypothesis 4.1: $T_{\text{down}} = 10$.

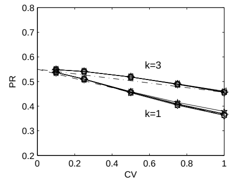
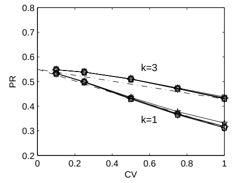
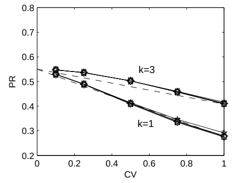
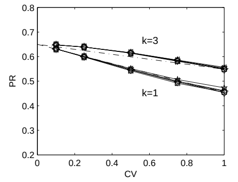
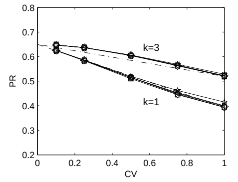
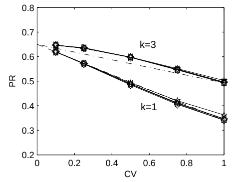
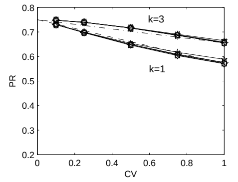
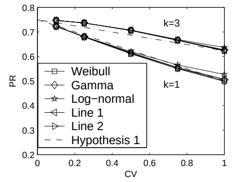
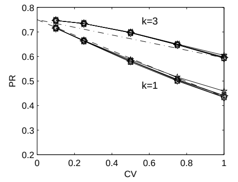
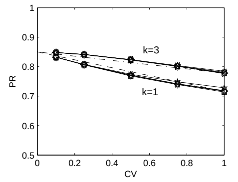
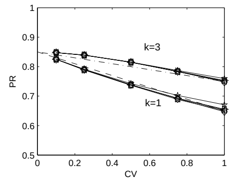
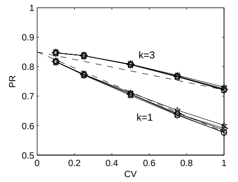
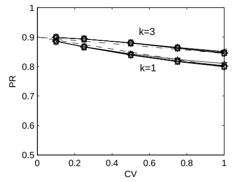
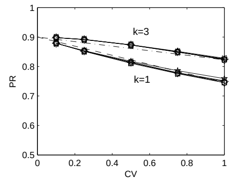
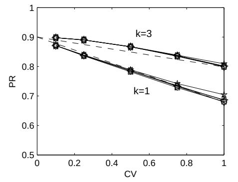
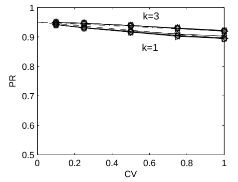
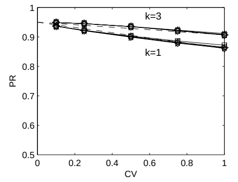
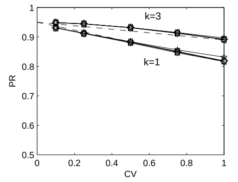
	M=3	M=5	M=10
e=0.55			
e=0.65			
e=0.75			
e=0.85			
e=0.9			
e=0.95			

Table 4.3. Production rates evaluated by simulations and by Hypothesis 4.1: $T_{\mathrm{down}} = 20$.

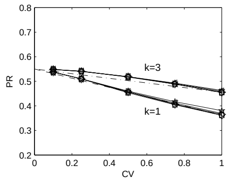
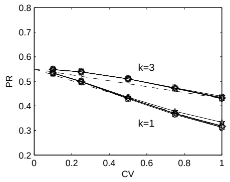
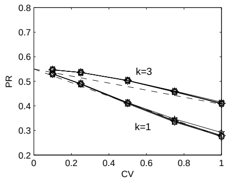
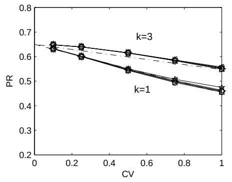
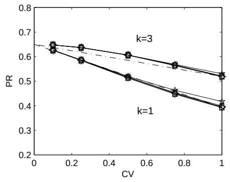
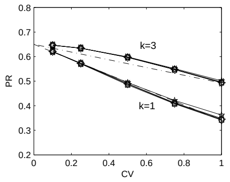
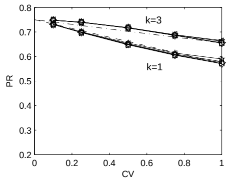
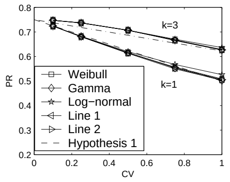
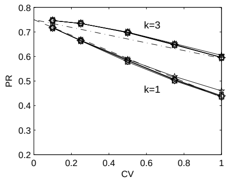
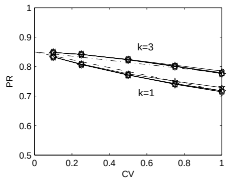
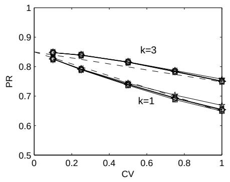
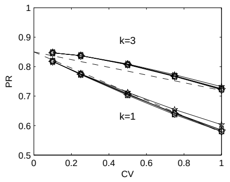
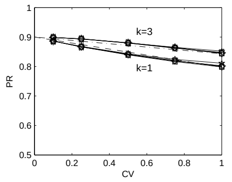
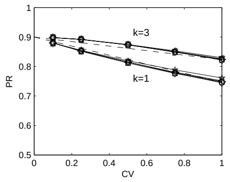
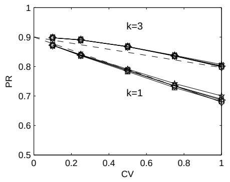
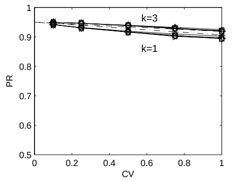
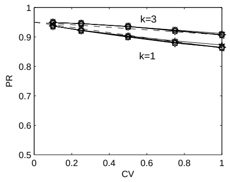
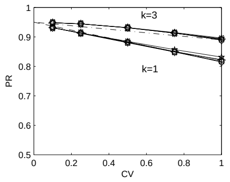
	M=3	M=5	M=10
e=0.55			
e=0.65			
e=0.75			
e=0.85			
e=0.9			
e=0.95			

Table 4.4. Sensitivity of production rate to the types of up- and downtime distributions (the values of ε calculated according to (4.17)) (a). $T_{\text{down}} = 10$.

		CV = 0.1			CV = 0.25			CV = 0.5		
	M	3	5	10	3	5	10	3	5	10
$e = 0.55$	$k = 1$	0.06	0.06	0.02	0.22	0.30	0.33	1.10	1.73	1.40
	$k = 3$	0.05	0.02	0.02	0.04	0.02	0.04	0.17	0.22	0
$e = 0.65$	$k = 1$	0.05	0.03	0.03	0.35	0.36	0.33	1.59	1.86	1.65
	$k = 3$	0.02	0.03	0.02	0.05	0.05	0	0.31	0.38	0.15
$e = 0.75$	$k = 1$	0.07	0.01	0.06	0.33	0.28	0.33	1.13	1.47	1.68
	$k = 3$	0.04	0.03	0	0.03	0.01	0.01	0.20	0.18	0.24
$e = 0.85$	$k = 1$	0.02	0.05	0.09	0.17	0.34	0.39	0.78	0.77	1.28
	$k = 3$	0.01	0.01	0.01	0.06	0.05	0.05	0.15	0.05	0.24
$e = 0.9$	$k = 1$	0.03	0.02	0.01	0.15	0.28	0.33	0.56	0.81	0.87
	$k = 3$	0.02	0.02	0	0.03	0.09	0.03	0.16	0.08	0.08
$e = 0.95$	$k = 1$	0.02	0.01	0.01	0.06	0.18	0.13	0.21	0.41	0.47
	$k = 3$	0.01	0	0.01	0.02	0.01	0.03	0.04	0.07	0.06

		CV = 0.75			CV = 1		
	M	3	5	10	3	5	10
$e = 0.55$	$k = 1$	2.93	3.54	3.68	4.50	6.25	6.08
	$k = 3$	0.47	0.74	0.68	1.21	1.77	1.64
$e = 0.65$	$k = 1$	3.11	3.87	3.75	4.44	5.54	6.37
	$k = 3$	0.84	1.05	1.01	1.66	1.56	2.09
$e = 0.75$	$k = 1$	2.20	2.91	3.02	3.19	5.63	5.86
	$k = 3$	0.63	0.57	0.06	1.44	2.25	1.97
$e = 0.85$	$k = 1$	1.29	2.00	2.69	2.00	3.65	4.29
	$k = 3$	0.35	0.47	0.69	1.03	1.62	1.43
$e = 0.9$	$k = 1$	0.97	1.30	1.87	1.38	1.84	3.62
	$k = 3$	0.36	0.27	0.50	0.75	0.64	1.26
$e = 0.95$	$k = 1$	0.40	0.87	1.17	0.54	1.23	1.79
	$k = 3$	0.14	0.26	0.26	0.22	0.60	0.59

Table 4.5. Sensitivity of production rate to the types of up- and downtime distributions (the values of ε calculated according to (4.17)) (b). $T_{\text{down}} = 20$.

		CV = 0.1			CV = 0.25			CV = 0.5		
	M	3	5	10	3	5	10	3	5	10
$e = 0.55$	$k = 1$	0.06	0.11	0.06	0.12	0.34	0.37	1.46	1.56	1.25
	$k = 3$	0.09	0.04	0.05	0.24	0.06	0.09	0.31	0.12	0.16
$e = 0.65$	$k = 1$	0.08	0.05	0.03	0.33	0.48	0.47	1.27	1.39	1.92
	$k = 3$	0.05	0.03	0.03	0	0.13	0.02	0.23	0.08	0.37
$e = 0.75$	$k = 1$	0.04	0	0.06	0.34	0.31	0.30	1.16	1.13	1.88
	$k = 3$	0.01	0.04	0.01	0.11	0.07	0.05	0.18	0.07	0.36
$e = 0.85$	$k = 1$	0.01	0.05	0.04	0.17	0.29	0.32	0.61	1.00	1.45
	$k = 3$	0.01	0.02	0.01	0.05	0.01	0.06	0.17	0.14	0.33
$e = 0.9$	$k = 1$	0.03	0.03	0.02	0.20	0.34	0.39	0.46	0.94	1.06
	$k = 3$	0.02	0.01	0.01	0.03	0.06	0.09	0.16	0.10	0.13
$e = 0.95$	$k = 1$	0.02	0.06	0.05	0.14	0.16	0.14	0.31	0.47	0.52
	$k = 3$	0.01	0.01	0.01	0.06	0.03	0.02	0.12	0.01	0.06

		CV = 0.75			CV = 1		
	M	3	5	10	3	5	10
$e = 0.55$	$k = 1$	3.12	3.68	3.74	5.34	6.70	6.05
	$k = 3$	0.86	0.62	0.94	1.91	4.02	1.66
$e = 0.65$	$k = 1$	2.69	3.65	3.38	4.22	6.27	5.94
	$k = 3$	0.55	0.80	0.75	1.64	2.18	1.62
$e = 0.75$	$k = 1$	1.47	2.77	3.45	3.44	4.39	5.43
	$k = 3$	0.32	0.81	0.90	1.67	1.61	1.49
$e = 0.85$	$k = 1$	1.42	2.27	2.67	2.10	3.04	4.00
	$k = 3$	0.54	0.58	0.80	0.09	1.04	1.43
$e = 0.9$	$k = 1$	0.93	1.52	1.89	1.57	2.23	3.06
	$k = 3$	0.36	0.45	0.30	0.75	1.00	1.04
$e = 0.95$	$k = 1$	0.64	0.88	0.98	1.01	1.00	2.07
	$k = 3$	0.30	0.39	0.24	0.65	0.47	0.81

$$k \uparrow \Rightarrow \text{ Sensitivity } \downarrow,$$
$$e \uparrow \Rightarrow \text{ Sensitivity } \downarrow.$$

(b) Hypothesis 4.1 (i.e., expression (4.8)) approximates well the production rate of all systems considered. Indeed, the values of metric Δ_1 (calculated according to (4.18) shown in Tables 4.6 and 4.7 are within 6%. Since this precision is commensurable with the accuracy of the data available on the factory floor with regard to machine and buffer parameters, we conclude that Hypothesis 4.1 can be used as a tool for evaluating the production rate in serial lines with identical machines obeying Weibull, gamma, and log-normal reliability models. We have also investigated the applicability of (4.8) for systems with Rayleigh and Erlang reliability models and obtained similar results. Based on this, we conjecture that expression (4.8) can be used for evaluating production rates in serial lines with identical machines obeying any reliability model, provided that buffer capacity is at least one downtime and probability density functions of up- and downtime are unimodal.

6. Production rate for the case of identical coefficients of variation

6.1 Parameters of systems analyzed

In the case of machines with arbitrary e_i and $T_{\mathrm{up},i}$ but with identical coefficients of variation, i.e.,

$$\mathrm{CV}_{\mathrm{up},i} = \mathrm{CV}_{\mathrm{up}}, \quad \mathrm{CV}_{\mathrm{down},i} = \mathrm{CV}_{\mathrm{down}}, \qquad i = 1, \ldots, M,$$

we consider lines (4.15) and (4.16) with parameters selected randomly and equiprobably from the sets:

$$T_{\mathrm{down},i} \in \{10, 20\}, \tag{4.20}$$
$$e_i \in \{0.85, 0.9, 0.95\} \tag{4.21}$$

or

$$e_i \in \{0.55, 0.65, 0.75, 0.85, 0.9, 0.95\}. \tag{4.22}$$

We use the sets (4.21) and (4.22) in order to investigate production lines with relatively efficient and relatively inefficient machines, respectively. As a results, the following parameter sets (PS) have been selected:

$$\begin{aligned} \text{PS1: } & T_{\mathrm{down}} = [20, 10, 10, 20, 20, 10, 20, 20, 10, 10], \\ & e = [0.95, 0.85, 0.9, 0.95, 0.85, 0.9, 0.95, 0.9, 0.9, 0.85], \qquad (4.23) \\ \text{PS2: } & T_{\mathrm{down}} = [10, 20, 20, 20, 10, 10, 20, 20, 10, 10], \end{aligned}$$

Table 4.6. Accuracy of Hypothesis 4.1 (the values of Δ_1 calculated according to (4.18)) (a). $T_{\text{down}} = 10$.

		CV = 0.1			CV = 0.25			CV = 0.5		
	M	3	5	10	3	5	10	3	5	10
$e = 0.55$	$k = 1$	1.14	1.33	1.34	1.36	1.76	1.72	0.61	1.10	1.00
	$k = 3$	1.40	1.82	2.09	2.64	3.51	4.09	3.04	4.22	4.67
$e = 0.65$	$k = 1$	0.17	0.16	0.08	0.40	0.38	0.42	1.79	2.23	2.28
	$k = 3$	1.22	1.58	1.90	2.27	3.06	4.03	2.70	3.56	4.40
$e = 0.75$	$k = 1$	0.33	0.35	0.43	1.18	1.24	1.27	2.18	2.36	2.61
	$k = 3$	0.96	1.31	1.62	1.78	2.48	3.15	2.01	2.91	3.74
$e = 0.85$	$k = 1$	0.53	0.63	0.79	1.30	1.43	1.40	1.91	1.78	1.64
	$k = 3$	0.62	0.90	1.20	1.14	1.72	2.33	1.18	1.91	2.81
$e = 0.9$	$k = 1$	0.48	0.65	0.81	0.98	1.17	1.06	1.25	1.46	0.99
	$k = 3$	0.45	0.66	0.87	0.86	1.29	1.76	0.97	1.41	2.10
$e = 0.95$	$k = 1$	0.36	0.49	0.62	0.63	0.75	0.55	0.76	0.71	0.35
	$k = 3$	0.22	0.34	0.49	0.40	0.64	1.01	0.40	0.74	1.27

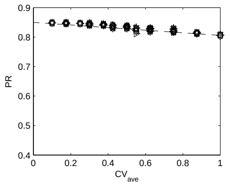
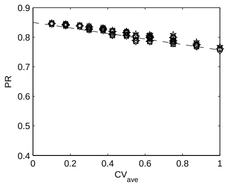
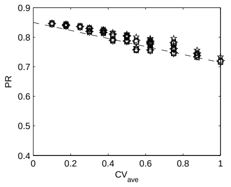
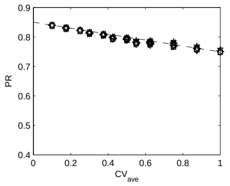
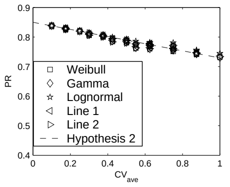
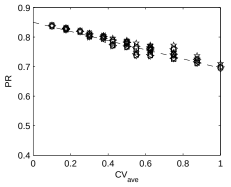
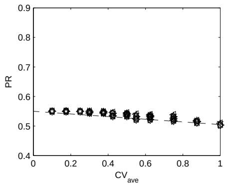
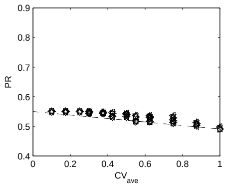
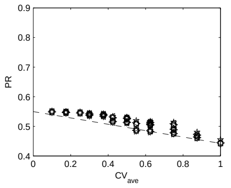
		CV = .75			CV = 1		
	M	3	5	10	3	5	10
$e = 0.55$	$k = 1$	1.52	1.86	2.99	4.23	5.85	5.59
	$k = 3$	2.20	3.21	3.49	0.91	2.01	1.76
$e = 0.65$	$k = 1$	2.36	2.79	3.19	4.04	5.28	5.88
	$k = 3$	2.05	2.85	3.45	1.46	1.65	1.95
$e = 0.75$	$k = 1$	1.92	2.36	2.17	3.06	5.03	5.16
	$k = 3$	1.64	2.14	2.72	1.35	2.04	1.72
$e = 0.85$	$k = 1$	1.40	1.81	1.71	1.68	3.13	3.76
	$k = 3$	0.95	1.54	2.24	0.83	1.29	1.30
$e = 0.9$	$k = 1$	0.91	1.23	0.99	1.52	1.79	3.51
	$k = 3$	0.80	1.14	1.74	0.82	0.65	1.33
$e = 0.95$	$k = 1$	0.55	0.52	0.80	0.46	1.27	1.89
	$k = 3$	0.30	0.61	1.09	0.25	0.49	0.75

Table 4.7. Accuracy of Hypothesis 4.1 (the values of Δ_1 calculated according to (4.18)) (b). $T_{\text{down}} = 20$.

		CV = 0.1			CV = 0.25			CV = 0.5		
	M	3	5	10	3	5	10	3	5	10
$e = 0.55$	$k = 1$	1.23	1.44	1.49	1.33	1.85	1.86	0.74	0.99	0.84
	$k = 3$	1.47	1.78	2.09	2.79	3.47	4.11	3.13	3.91	5.06
$e = 0.65$	$k = 1$	0.35	0.32	0.27	0.17	0.31	0.14	1.45	1.83	2.14
	$k = 3$	1.34	1.64	1.90	2.42	3.12	3.78	2.76	3.58	4.59
$e = 0.75$	$k = 1$	0.23	0.14	0.24	1.10	1.06	1.07	2.07	1.99	2.46
	$k = 3$	1.00	1.35	1.62	1.84	2.56	3.28	2.09	2.93	3.98
$e = 0.85$	$k = 1$	0.45	0.44	0.61	1.21	1.26	1.15	1.62	1.66	1.57
	$k = 3$	0.65	0.94	1.19	1.16	1.82	2.42	1.25	2.07	3.03
$e = 0.9$	$k = 1$	0.42	0.58	0.69	0.93	1.14	0.98	1.12	1.43	0.88
	$k = 3$	0.49	0.66	0.89	0.93	1.29	1.87	1.02	1.48	2.28
$e = 0.95$	$k = 1$	0.31	0.48	0.58	0.58	0.72	0.48	0.59	0.76	0.34
	$k = 3$	0.25	0.35	0.45	0.46	0.66	1.00	0.58	0.82	1.25

		CV = .75			CV = 1		
	M	3	5	10	3	5	10
$e = 0.55$	$k = 1$	1.25	1.99	2.64	4.65	6.25	5.94
	$k = 3$	2.76	2.77	3.94	1.64	1.91	2.09
$e = 0.65$	$k = 1$	1.58	2.46	2.55	4.47	5.52	5.88
	$k = 3$	2.30	2.89	3.50	1.90	2.07	1.81
$e = 0.75$	$k = 1$	1.82	1.97	2.30	3.12	5.04	5.72
	$k = 3$	1.38	2.37	3.13	1.52	2.12	1.98
$e = 0.85$	$k = 1$	1.36	1.51	1.36	1.67	3.15	4.28
	$k = 3$	1.07	1.73	2.66	0.82	1.27	1.73
$e = 0.9$	$k = 1$	0.83	1.06	1.01	1.64	1.93	3.00
	$k = 3$	0.90	1.29	1.85	1.06	0.82	1.30
$e = 0.95$	$k = 1$	0.67	0.58	0.84	1.00	1.19	1.84
	$k = 3$	0.66	0.67	1.10	0.67	0.56	0.69

Table 4.8. Production rates evaluated by simulation and Hypothesis 4.2.

	$M = 3$	$M = 5$	$M = 10$
PS1			
PS2			
PS3			

$$e = [0.9, 0.85, 0.85, 0.95, 0.9, 0.95, 0.9, 0.85, 0.9, 0.95], \qquad (4.24)$$

$$\text{PS3}: T_{\text{down}} = [10, 20, 10, 20, 10, 20, 20, 10, 10, 20],$$

$$e = [0.55, 0.75, 0.9, 0.85, 0.65, 0.9, 0.55, 0.95, 0.65, 0.75], \qquad (4.25)$$

where T_{down} and e are vectors with elements $T_{\text{down},i}$ and e_i, $i = 1, \ldots, 10$, respectively.

Each of these parameter sets is used to calculate PR in all lines (4.15), (4.16) with $M \in \{3, 5, 10\}$, $k = 1$, CV_{up} and $\text{CV}_{\text{down}} \in \{0.1,\ 0.25,\ 0.5,\ 0.75,\ 1\}$. Thus, the total of 1125 production lines have been analyzed.

6.2 Results

Table 4.8 presents the simulation results for all systems analyzed along with the dashed line corresponding to Hypothesis 4.2, i.e.,

$$\text{PR} = e_{\min} - (e_{\min} - \text{PR}^{\exp})\text{CV}_{\text{ave}},$$

where

$$\text{CV}_{\text{ave}} = \frac{\text{CV}_{\text{up}} + \text{CV}_{\text{down}}}{2}.$$

Table 4.9 characterizes the accuracy of Hypothesis 4.2 (i.e., metric Δ_2 defined in (4.19)). As one can see, Δ_2 is within 6%. Thus, we

Table 4.9. Accuracy of Hypothesis 4.2 (the values of Δ_2 calculated according to (4.19)).

$\mathrm{CV}_{\mathrm{ave}}$		0.1	0.175	0.25	0.3	0.375	0.425
PS1	$M = 3$	0.34	0.73	0.87	1.21	1.25	1.12
	$M = 5$	0.64	1.16	1.44	1.61	1.90	1.73
	$M = 10$	1.12	1.91	2.37	2.57	3.02	2.91
PS2	$M = 3$	0.13	0.44	0.45	1.06	0.93	1.76
	$M = 5$	0.06	0.32	0.17	0.99	0.53	1.91
	$M = 10$	0.47	0.81	1.00	1.22	1.59	1.70
PS3	$M = 3$	0.82	1.43	1.97	2.43	2.80	2.76
	$M = 5$	1.07	1.88	2.65	3.13	3.67	3.90
	$M = 10$	1.82	3.05	4.17	4.68	5.66	5.43

$\mathrm{CV}_{\mathrm{ave}}$		0.5	0.55	0.625	0.75	0.875	1
PS1	$M = 3$	1.37	1.67	1.23	1.02	0.65	0.81
	$M = 5$	2.06	1.62	2.21	2.20	2.07	1.64
	$M = 10$	3.50	3.17	3.86	3.81	3.27	2.55
PS2	$M = 3$	1.28	2.35	1.69	1.20	1.04	1.27
	$M = 5$	1.21	2.73	1.93	1.13	1.55	2.07
	$M = 10$	2.15	3.51	2.56	3.05	3.06	2.15
PS3	$M = 3$	3.10	2.79	2.99	2.56	1.86	1.35
	$M = 5$	4.21	3.96	4.25	4.30	3.28	2.04
	$M = 10$	6.16	5.52	6.58	6.35	5.11	2.72

conjecture that the production rate for systems with identical coefficients of variation and any distribution of up- and downtime can be evaluated using Hypothesis 4.2 (i.e., expression (4.9)).

REMARK 10 In Table 4.8, some of the $\mathrm{CV}_{\mathrm{ave}}$ correspond to multiple values of PR. This is because different selection of $\mathrm{CV}_{\mathrm{up}}$ and $\mathrm{CV}_{\mathrm{down}}$ may result in same $\mathrm{CV}_{\mathrm{ave}}$. For instance, $\{\mathrm{CV}_{\mathrm{up}} = 0.25, \mathrm{CV}_{\mathrm{down}} = 0.75\}$, $\{\mathrm{CV}_{\mathrm{up}} = \mathrm{CV}_{\mathrm{down}} = 0.5\}$, and $\{\mathrm{CV}_{\mathrm{up}} = 0.75, \mathrm{CV}_{\mathrm{down}} = 0.25\}$ all lead to $\mathrm{CV}_{\mathrm{ave}} = 0.5$. As one can see, the differences among all values of PR corresponding to the same $\mathrm{CV}_{\mathrm{ave}}$ are not significant, and Hypothesis 4.2 approximates well to all values of PR. This again verifies that the PR depends mostly on the first two moments of up- and downtime distribution.

7. Production rate evaluation for the general case

7.1 Parameters of systems analyzed

In the general case, we again consider production lines (4.15) and (4.16) and parameter sets (4.23)–(4.25) but with $\mathrm{CV}_{\mathrm{up},i}$ and $\mathrm{CV}_{\mathrm{down},i}$ selected randomly. Specifically, we select $\mathrm{CV}_{\mathrm{up},i}$ and $\mathrm{CV}_{\mathrm{down},i}$ using the following probability mass functions defined on $\{0.1, 0.25, 0.5, 0.75, 1\}$:

- uniform distribution, i.e., both $\mathrm{CV}_{\mathrm{up}}$ and $\mathrm{CV}_{\mathrm{down}}$ are selected equiprobably;
- increasing triangular distribution, i.e., both $\mathrm{CV}_{\mathrm{up}}$ and $\mathrm{CV}_{\mathrm{down}}$ are selected according to triangular distributions with higher values of CVs being selected with larger probabilities than lower values;
- decreasing triangular distribution, i.e., both $\mathrm{CV}_{\mathrm{up}}$ and $\mathrm{CV}_{\mathrm{down}}$ are selected according to triangular distributions with lower values of CVs being selected with larger probabilities than higher values;
- increasing/decreasing triangular distributions, i.e., $\mathrm{CV}_{\mathrm{up}}$ (respectively, $\mathrm{CV}_{\mathrm{down}}$) is selected according to triangular distributions where higher values (respectively, lower values) are more probable than lower (respectively, higher) values.

As a result, we obtained the following sequences (S) of $\mathrm{CV}_{\mathrm{up}}$ and $\mathrm{CV}_{\mathrm{down}}$:

$$\begin{aligned}
\mathrm{S1}: \quad & \mathrm{CV}_{\mathrm{up}} = [1, 0.25, 0.75, 0.5, 1, 0.75, 0.5, 0.1, 1, 0.5], \\
& \mathrm{CV}_{\mathrm{down}} = [0.25, 0.5, 0.75, 0.75, 0.25, 0.5, 1, 0.1, 0.75, 0.1], && (4.26) \\
\mathrm{S2}: \quad & \mathrm{CV}_{\mathrm{up}} = [1, 0.5, 1, 0.75, 0.25, 1, 0.75, 1, 1, 1], \\
& \mathrm{CV}_{\mathrm{down}} = [1, 1, 0.25, 1, 0.75, 1, 1, 0.75, 1, 0.5], && (4.27) \\
\mathrm{S3}: \quad & \mathrm{CV}_{\mathrm{up}} = [0.1, 0.25, 0.1, 0.5, 0.1, 0.1, 0.1, 0.75, 0.1, 0.1], \\
& \mathrm{CV}_{\mathrm{down}} = [0.75, 0.1, 0.25, 0.1, 0.5, 0.1, 0.1, 0.25, 0.1, 0.1], && (4.28) \\
\mathrm{S4}: \quad & \mathrm{CV}_{\mathrm{up}} = [0.25, 0.1, 0.1, 0.1, 0.25, 0.1, 0.1, 0.5, 0.1, 0.1], \\
& \mathrm{CV}_{\mathrm{down}} = [0.1, 0.25, 0.1, 0.1, 0.1, 0.1, 0.5, 0.1, 0.1, 0.1], && (4.29) \\
\mathrm{S5}: \quad & \mathrm{CV}_{\mathrm{up}} = [0.75, 1, 1, 0.75, 1, 1, 1, 0.5, 1, 1], \\
& \mathrm{CV}_{\mathrm{down}} = [1, 1, 0.75, 1, 1, 1, 0.5, 1, 1, 1], && (4.30) \\
\mathrm{S6}: \quad & \mathrm{CV}_{\mathrm{up}} = [1, 0.75, 0.5, 1, 1, 0.25, 1, 0.75, 1, 0.1], \\
& \mathrm{CV}_{\mathrm{down}} = [0.1, 0.25, 0.5, 0.1, 0.1, 0.75, 0.1, 1, 0.1, 0.1], && (4.31) \\
\mathrm{S7}: \quad & \mathrm{CV}_{\mathrm{up}} = [0.1, 0.25, 0.1, 0.1, 0.5, 0.1, 1, 0.1, 0.1, 0.75], \\
& \mathrm{CV}_{\mathrm{down}} = [1, 0.5, 0.75, 1, 0.5, 1, 0.75, 0.25, 0.1, 1], && (4.32)
\end{aligned}$$

where $\mathrm{CV}_{\mathrm{up}}$ and $\mathrm{CV}_{\mathrm{down}}$ are vectors with components $\mathrm{CV}_{\mathrm{up},i}$ and $\mathrm{CV}_{\mathrm{down},i}$, $i = 1, \ldots, 10$, respectively.

Each of these sequences is used for all systems (4.15), (4.16) with parameter sets (4.23)–(4.25), $M \in \{3, 5, 10\}$ and $k = 1$. Thus, the total of 315 systems have been analyzed.

7.2 Results

Table 4.10 presents the results for all systems analyzed by simulations and by Hypothesis 4.3, i.e.,

$$\mathrm{PR} = e_{\min} - (e_{\min} - \mathrm{PR}^{\exp})\mathrm{CV}_{\mathrm{ave}},$$

where

$$\mathrm{CV}_{\mathrm{ave}} = \sum_{i=1}^{M} \frac{\mathrm{CV}_{\mathrm{up},i} + \mathrm{CV}_{\mathrm{down},i}}{2M},$$

represented by the broken line. Table 4.11 characterizes the accuracy of Hypothesis 4.3 by showing the values of Δ_3. Clearly, this accuracy is again within 4%. Thus, we conjecture that expression (4.10) can be used for evaluating the throughput in serial production lines with arbitrary unimodal distributions of up- and downtime.

8. Conclusions

The method developed in this paper can be used as a tool for a quick analytic evaluation of the performance of serial production lines with arbitrary models of machines reliability. Its application does not require more information than that necessary for such an evaluation under the exponential assumption. Indeed, since under the exponential assumption one needs the data to evaluate the average up- and downtime of the machines, the same data can be used to evaluate their coefficients of variation. Thus, all the information necessary for (4.8)–(4.10) is available, and the production rate may be evaluated without the knowledge of the types of up- and downtime distributions. This is the main advantage of the method developed.

The drawbacks of this method are as follows:

1. All buffers must be large enough to accommodate at least one largest downtime of all machines in the system.
2. No analytical proofs of relationships (4.8)–(4.10) are available.

These drawbacks can be commented upon as follows:

Table 4.10. Production rates evaluated by simulation and Hypothesis 4.3.

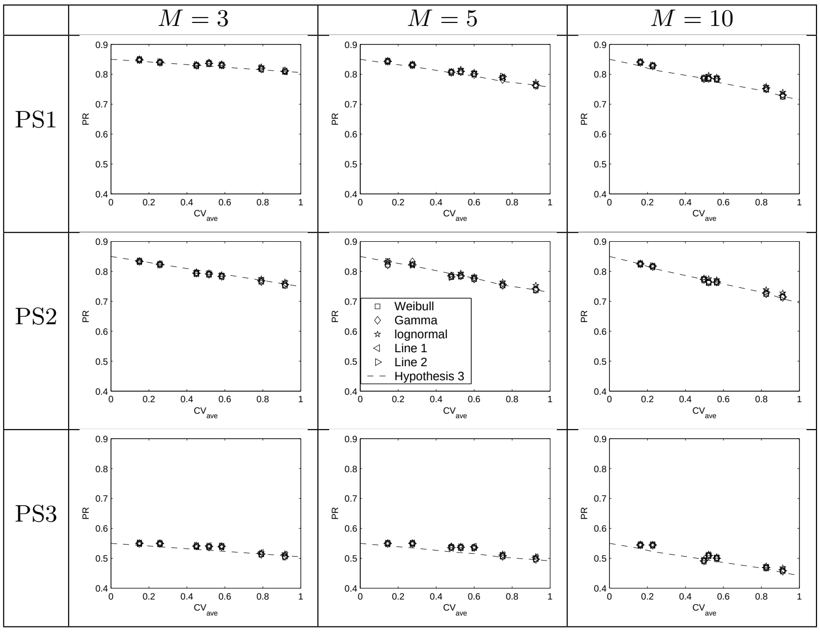


Table 4.11. Accuracy of Hypothesis 4.3 (the values of Δ_3 calculated according to (4.19)).

$M = 3$	CV$_{\text{ave}}$	0.15	0.2583	0.45	0.5167	0.5833	0.7917	0.9167
	PS1	0.60	0.26	0.22	1.51	1.22	1.18	0.47
	PS2	0.18	0.16	1.62	1.12	1.03	0.61	0.93
	PS3	1.22	2.00	1.96	2.61	3.00	0.85	1.22
$M = 5$	CV$_{\text{ave}}$	0.145	0.275	0.48	0.53	0.6	0.75	0.925
	PS1	0.82	0.67	0.45	1.98	1.59	2.65	1.23
	PS2	0.05	0.47	1.40	0.98	0.60	1.28	1.95
	PS3	1.55	2.73	3.35	3.93	3.81	2.12	1.94
$M = 10$	CV$_{\text{ave}}$	0.1625	0.2275	0.4975	0.5225	0.565	0.825	0.9225
	PS1	2.45	1.77	0.70	2.20	2.10	2.19	1.81
	PS2	0.06	0.93	0.09	0.97	1.24	1.53	2.64
	PS3	2.28	4.09	1.12	3.89	3.17	1.98	3.34

We believe that the method developed in this paper can be extended to production lines with buffers smaller than those necessary for accommodating the largest downtime of all the machines. Since in this case Theorem 4.1 does not hold, to evaluate the production rate for $\mathrm{CV}_{\mathrm{up}} = \mathrm{CV}_{\mathrm{down}} = 0$ simulations would be necessary. An investigation of such an extension is a topic for future work.

As for the lack of analytical proofs of (4.8)–(4.10), we believe that, at present, they are all but impossible.

Appendix: Proof of Theorem 4.1

The proof of Theorem 4.1 is based on the following two lemmas:

LEMMA 4.1 *Consider a two-machine line defined by assumptions* (i)–(v) *with* $\mathrm{CV}_{\mathrm{up},i} = \mathrm{CV}_{\mathrm{down},i} = 0$, $i = 1, 2$, *and assume that* $T_{\mathrm{up},1} = T_{\mathrm{up},2} = T_{\mathrm{up}}$, $T_{\mathrm{down},1} = T_{\mathrm{down},2} = T_{\mathrm{down}}$. *Then its production rate is given by*

$$\mathrm{PR} = e = \frac{T_{\mathrm{up}}}{T_{\mathrm{up}} + T_{\mathrm{down}}}.$$

Proof. Let $s_i(t)$, $i = 1, 2$ denote the states of machine m_i, $i = 1, 2$, at time t,

$$s_i(t) = \begin{cases} 0, & m_i \text{ is down,} \\ 1, & m_i \text{ is up,} \end{cases}$$

and $h(t)$ be the buffer occupancy at time t. Without loss of generality, $h(0) = 0$.

Assume that $s_i(0) = 0$, $i = 1, 2$, and m_i changes its state at time $t_i \leq T_{\mathrm{down}}$, $i = 1, 2$, i.e.,

$$s_i(t) = \begin{cases} 0, & t \in [0, t_i), \\ 1, & t \in [t_i + lT, t_i + lT + T_{\mathrm{up}}), \\ 0, & t \in [t_i + lT + T_{\mathrm{up}}, t_i + (l+1)T), \\ & i = 1, 2, l = 0, 1, 2, \ldots, \end{cases}$$

where $T = T_{\mathrm{up}} + T_{\mathrm{down}}$. Then, the following three cases are possible:

Case 1. $t_1 = t_2$. Obviously, in this case machine m_1 is never blocked and m_2 is never starved, and therefore, $\mathrm{PR} = e$.

Case 2. $t_1 > t_2$. In this case,

(a) If $t_2 + T_{\mathrm{up}} \geq t_1$, the following hold:

$$\begin{array}{ll}
s_1(t)=0,\, s_2(t)=0,\, h(t)=0, & t \in [0, t_2), \\
s_1(t)=0,\, s_2(t)=1,\, h(t)=0, & t \in [t_2, t_1), \\
s_1(t)=1,\, s_2(t)=1,\, h(t)=0, & t \in [t_1, t_2 + T_{\mathrm{up}}), \\
s_1(t)=1,\, s_2(t)=0,\, h(t)=t - T_{\mathrm{up}} - t_2, & t \in [t_2 + T_{\mathrm{up}}, t_1 + T_{\mathrm{up}}), \\
s_1(t)=0,\, s_2(t)=0,\, h(t)=t_1 - t_2, & t \in [t_1 + T_{\mathrm{up}}, t_2 + T), \\
s_1(t)=0,\, s_2(t)=1,\, h(t)=t_1 + T - t, & t \in [t_2 + T, t_1 + T), \\
s_1(t)=1,\, s_2(t)=1,\, h(t)=0, & t \in [t_1 + T, t_2 + T + T_{\mathrm{up}}), \\
s_1(t)=1,\, s_2(t)=0,\, h(t)=t - t_2 - T - T_{\mathrm{up}}, & t \in [t_2 + T + T_{\mathrm{up}}, t_1 + T + T_{\mathrm{up}}), \\
s_1(t)=0,\, s_2(t)=0,\, h(t)=t_1 - t_2, & t \in [t_1 + T + T_{\mathrm{up}}, t_2 + 2T), \\
\ldots,\quad \ldots,\quad \ldots, & \ldots.
\end{array}$$

By induction, we obtain

$$\begin{array}{ll}
s_1(t)=0,\, s_2(t)=0,\, h(t)=t_1 - t_2, & t \in [t_1 + (l-1)T + T_{\mathrm{up}}, t_2 + lT), \\
s_1(t)=0,\, s_2(t)=1,\, h(t)=t_1 - t + lT, & t \in [t_2 + lT, t_1 + lT), \\
s_1(t)=1,\, s_2(t)=1,\, h(t)=0, & t \in [t_1 + lT, t_2 + lT + T_{\mathrm{up}}), \\
s_1(t)=1,\, s_2(t)=0,\, h(t)=t - t_2 - lT - T_{\mathrm{up}}, & t \in [t_2 + lT + T_{\mathrm{up}}, t_1 + lT + T_{\mathrm{up}}), \\
\qquad l=1,2,\ldots. &
\end{array}$$

Since $k \geq 1$, i.e., $N \geq T_{\mathrm{down}}$, machine m_1 is never blocked and machine m_2 is not starved (except for the initial period $t \in [t_2, t_1)$). Thus, after the period $[0, t_2 + T_{\mathrm{up}})$, the line is producing during the interval of length T_{up} and is not producing during the interval of length T_{down}.

(b) If $t_2 + T_{\mathrm{up}} < t_1$, which implies that $T_{\mathrm{up}} < T_{\mathrm{down}}$, the following hold:

$$\begin{array}{ll}
s_1(t)=0,\, s_2(t)=0,\, h(t)=0, & t \in [0, t_2), \\
s_1(t)=0,\, s_2(t)=1,\, h(t)=0, & t \in [t_2, t_2 + T_{\mathrm{up}}), \\
s_1(t)=0,\, s_2(t)=0,\, h(t)=0, & t \in [t_2 + T_{\mathrm{up}}, t_1), \\
s_1(t)=1,\, s_2(t)=0,\, h(t)=t - t_1, & t \in [t_1, t_1 + T_{\mathrm{up}}), \\
s_1(t)=0,\, s_2(t)=0,\, h(t)=T_{\mathrm{up}}, & t \in [t_1 + T_{\mathrm{up}}, t_2 + T), \\
s_1(t)=0,\, s_2(t)=1,\, h(t)=T_{\mathrm{up}} - t + t_2 + T, & t \in [t_2 + T, t_2 + T + T_{\mathrm{up}}), \\
s_1(t)=0,\, s_2(t)=0,\, h(t)=0, & t \in [t_2 + T + T_{\mathrm{up}}, t_1 + T), \\
s_1(t)=1,\, s_2(t)=1,\, h(t)=t - t_1 - T, & t \in [t_1 + T, t_1 + T + T_{\mathrm{up}}), \\
\ldots,\quad \ldots,\quad \ldots, & \ldots.
\end{array}$$

Clearly, in this case,

$$\begin{array}{ll}
s_1(t)=0,\, s_2(t)=0,\, h(t)=0, & t \in [t_2 + lT + T_{\mathrm{up}}, t_1 + lT), \\
s_1(t)=1,\, s_2(t)=0,\, h(t)=t - t_1 - kT, & t \in [t_1 + lT, t_1 + lT + T_{\mathrm{up}}), \\
s_1(t)=0,\, s_2(t)=0,\, h(t)=T_{\mathrm{up}}, & t \in [t_1 + lT + T_{\mathrm{up}}, t_2 + (l+1)T), \\
s_1(t)=0,\, s_2(t)=1,\, h(t)=t_2 + (l+1)T + T_{\mathrm{up}} - t, & t \in [t_2 + (l+1)T, t_2 + (l+1)T + T_{\mathrm{up}}), \\
\qquad l=1,2,\ldots. &
\end{array}$$

Thus, m_2 is not starved $\big($except for the initial interval $t \in [t_2, t_2 + T_{\mathrm{up}})\big)$ and m_1 is never blocked.

Therefore, in both (a) and (b), $\mathrm{PR} = e$.

Case 3. $t_1 < t_2$**.** Similar arguments can be used. If $t_1 + T_{\mathrm{up}} \geq t_2$, we show that:

$$\begin{array}{lll}
s_1(t)=0, s_2(t)=0, & h(t)=0, & t \in [t_2 + (l-1)T + T_{\mathrm{up}}, t_1 + lT),\\
s_1(t)=1, s_2(t)=0, & h(t)=t - t_1 - lT, & t \in [t_1 + lT, t_2 + lT),\\
s_1(t)=1, s_2(t)=1, & h(t)=t_2 - t_1, & t \in [t_2 + lT, t_1 + lT + T_{\mathrm{up}}),\\
s_1(t)=0, s_2(t)=1, & h(t)=t_2 - t + lT + T_{\mathrm{up}}, & t \in [t_1 + lT + T_{\mathrm{up}}, t_2 + lT + T_{\mathrm{up}}),\\
 & l=1,2,\ldots. &
\end{array}$$

Again, due to $N \geq T_{\mathrm{down}}$, machine m_1 is never blocked and m_2 is never starved. If $t_1 + T_{\mathrm{up}} < t_2$, the following hold:

$$\begin{array}{lll}
s_1(t)=0, s_2(t)=0, & h(t)=0, & t \in [t_2 + (l-1)T + T_{\mathrm{up}}, t_1 + lT),\\
s_1(t)=1, s_2(t)=0, & h(t)=t - t_1 - lT, & t \in [t_1 + lT, t_1 + lT + T_{\mathrm{up}}),\\
s_1(t)=0, s_2(t)=0, & h(t)=T_{\mathrm{up}}, & t \in [t_1 + lT + T_{\mathrm{up}}, t_2 + lT),\\
s_1(t)=0, s_2(t)=1, & h(t)=T_{\mathrm{up}} - t + lT + t_2, & t \in [t_2 + lT, t_2 + lT + T_{\mathrm{up}}),\\
 & l=1,2,\ldots. &
\end{array}$$

Therefore, in both cases, $\mathrm{PR} = e$.

Next, we repeat this analysis under the assumptions

$$\begin{array}{ll}
(\alpha) s_i(0) = 1, & i = 1, 2,\\
(\beta) s_1(0) = 1, & s_2(0) = 0,\\
(\gamma) s_1(0) = 0, & s_2(0) = 1,
\end{array}$$

and show that after the initial period $t \in [0, T_{\mathrm{up}} + T_{\mathrm{down}})$, the line produces during the interval of the length T_{up} and does not produce during the interval of the length T_{down}. Therefore, in all cases, $\mathrm{PR} = e$. □

Lemma 4.2 *Consider an M-machine line defined by assumption* (i)–(v) *with* $\mathrm{CV}_{\mathrm{up},i} = \mathrm{CV}_{\mathrm{down},i} = 0$, $i = 1, \ldots, M$, *and assume that* $T_{\mathrm{up},i} = T_{\mathrm{up}}$ *and* $T_{\mathrm{down},i} = T_{\mathrm{down}}$, $i = 1, \ldots, M$. *Then its production rate is given by*

$$\mathrm{PR} = e = \frac{T_{\mathrm{up}}}{T_{\mathrm{up}} + T_{\mathrm{down}}}.$$

Proof. From Lemma 4.1, machine m_2 is not starved except for a subinterval of $[0, T_{\mathrm{up}} + T_{\mathrm{down}})$. Thus, in the steady state, the two-machine line is up (producing) for T_{up} time units and down (not producing) for T_{down}, which is equivalent to a single machine. Since $N \geq T_{\mathrm{down}}$, no blockage of m_2 can occur. Therefore, aggregating these two machines with the third one, we conclude that the three machines are again equivalent to a single machine. Continuing this process until all machines are aggregated, we obtain $\mathrm{PR} = e$ for M-machine line when $\mathrm{CV} = 0$ and $k \geq 1$. □

Proof of Theorem 4.1. Consider the production line, L, defined by assumptions (i)–(v) with arbitrary values of $T_{\mathrm{up},i}$ and $T_{\mathrm{down},i}$, but with $\mathrm{CV}_{\mathrm{up},i} = \mathrm{CV}_{\mathrm{down},i}=0$, $i = 1, \ldots, M$. Along with it, consider the production line, L', also defined by (i)–(v) but with identical machines and buffers given by:

$$\begin{aligned} T'_{\text{down}} &= \max_{i=1,\ldots,M} T_{\text{down},i}, \\ e' &= \min_{i=1,\ldots,M} \frac{T_{\text{up},i}}{T_{\text{up}} + T_{\text{down},i}} = \min_{i=1,\ldots,M} e_i = e_{\min}, \\ T'_{\text{up}} &= \frac{e'}{1-e'} T'_{\text{down}}, \\ N' &= T'_{\text{down}}. \end{aligned}$$

As it follows from Lemma 4.2, the production rate, PR$'$, of line L' is $e_{\min}$. Due to (4.A.1) and the monotonicity property of production rate of serial lines with respect to machine and buffer parameters (see Shanthikumar and Yao, 1989), the production rate, PR, of line L satisfies the inequality

$$\text{PR} \geq \text{PR}'.$$

However, PR is limited by the least efficient machine in the system. Therefore,

$$\text{PR} = \text{PR}' = e_{\min}. \qquad \square$$

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