

Closed Bernoulli Production Lines: Analysis, Continuous Improvement, and Leanness

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Abstract—In closed production lines, each part is placed on a carrier at the input of the first machine and is removed from the carrier at the output of the last machine. The first machine is starved if no carriers are available, and the last machine is blocked if the empty carrier buffer is full. The number of carriers in the system is S and the capacity of the empty carrier buffer is N_0 . Under the assumption that the machines obey the Bernoulli reliability model, this paper provides methods for determining if a pair (N_0, S) impedes the open line performance and, if it does, develops techniques for improvement with respect to S and N_0 . In addition, bottlenecks in closed lines are discussed, and an approach to selecting the smallest N_0 and S , which result in no impediment, is described.

Note to Practitioners—In closed production lines, parts are transported throughout the system on pallets. Clearly, this could impede the system performance since the first operation may be starved for pallets and the last may be blocked by full empty pallets buffer. Therefore, it is important to establish conditions on the number of pallets and the capacity of the empty pallets buffer, under which the impediment does not take place. This paper provides such conditions under the assumption that the operations obey a simple (Bernoulli) reliability model. In addition, this paper provides a method for identification of bottlenecks in closed lines. Future work will extend these results to systems with exponential and non-Markovian reliability models.

Index Terms—Bernoulli machines, bottleneck identification, closed production line.

I. INTRODUCTION

PRODUCTION lines in large volume manufacturing environment often have parts transported from one operation to another on carriers (sometimes referred to as pallets, skids, etc.). Since in this situation the number of parts in the system is bounded by the number of available carriers, these lines are called *closed with respect to carriers* (or just *closed*). An example of such a line with M machines is given in Fig. 1, where the empty carrier buffer, b_0 , has the capacity N_0 and the number

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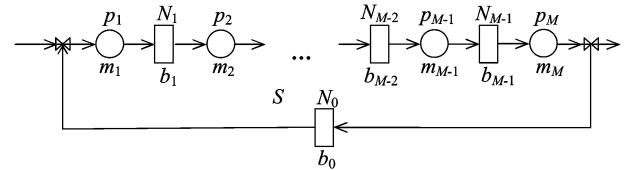


Fig. 1. Closed serial production line.

of carriers in the system is S . For the purposes of this paper, we assume that the machines obey the Bernoulli reliability model [1], i.e., when not starved or blocked, machine m_i produces a part during a cycle time with probability p_i and fails to do so with probability $1 - p_i$, $i = 1, \dots, M$ (see Section II for a precise model formulation). Such a reliability model is appropriate for assembly and painting operations where machines' downtime is comparable with the cycle time. The in-process buffers, b_i , $i = 1, \dots, M - 1$, are assumed to be of capacity $N_i < \infty$ and, therefore

$$S \leq \sum_{i=0}^{M-1} N_i. \quad (1)$$

Since in a closed line, the first machine can be starved for carriers and the last blocked by full empty carrier buffer, the production rate of the closed line, PR_{cl} , is, at best, equal to that of the corresponding open line, PR_o . If, however, either N_0 or S or both are chosen inappropriately, the closed nature of the line impedes the system performance and, as a result, PR_{cl} can be substantially lower than PR_o . An illustration is given in Fig. 2, where PR_{cl} (calculated using the methods of Section III) is shown as a function of S and N_0 for closed lines with two and five identical machines; PR_o is also indicated in Fig. 2 by broken lines. These graphs can be interpreted as follows: For the system of Fig. 2(a), the empty carrier buffer capacity $N_0 = 2$ is too small, since $PR_{cl} < PR_o$ for any S . With $N_0 = 4$, there is a single value of S (specifically, $S = 4$), which guarantees $PR_{cl} = PR_o$. When $N_0 = 6$, the equality $PR_{cl} = PR_o$ holds for $S \in \{4, 5, 6\}$. Finally, when $N_0 = 10$, the set of "non-impeding" S 's becomes even larger ($S \in \{4, 5, 6, 7, 8, 9, 10\}$). Clearly, the drop of PR_{cl} for small and for large values of S is due to starvation of m_1 for carriers and blockage of m_2 by empty carrier buffer, respectively. In addition, Fig. 2(a) shows that PR_{cl} is practically (however, not exactly) symmetric in S . A similar interpretation can be given for Fig. 2(b) as well.

Given the above, a question arises: How should N_0 and S be selected so that, on one hand, the closed nature of the line does not impede the open line performance and, on the other hand,

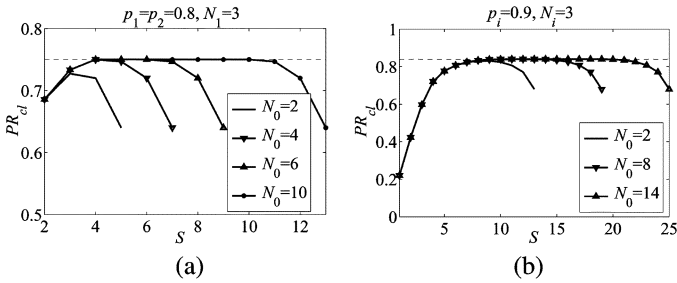


Fig. 2. Production rate as a function of the number of carriers and empty carrier buffer capacity. (a) Two-machine line. (b) Five-machine line.

N_0 and S are sufficiently small so that the closed line is “lean”? This is the question addressed in this paper.

Although closed lines have been discussed in the literature for almost five decades (see a review article [2]), the question posed above does not have a complete answer, and the function $PR_{cl}(N_0, S)$ has not been investigated in details. Nevertheless, a number of important results have been obtained. Specifically, methods for evaluating PR_{cl} have been developed in [3]–[8]. Various structural properties of PR_{cl} have been investigated in [9]. The equivalence of closed and appropriately defined open two-machine lines has been established in [10] and [11]. Important results concerning the value of S that maximizes PR_{cl} have been reported in [12]. Performance evaluation and control of multi-loop production systems are discussed in [13]. Finally, [14] and [15] provide results on buffer capacity allocation in closed lines.

The main contributions of this paper are as follows.

- Method for evaluating performance measures for two-machine closed lines is developed and a more restrictive result for $M > 2$ -machine lines is given.
- Notion of unimpeding pair (N_0, S) is introduced and criteria of unimpediment are established.
- Notions of improvability with respect to S and N_0 are introduced and criteria for determining when they take place are derived. Based on these criteria, procedures for continuous improvement with respect to S and N_0 are formulated.
- Method for identifying bottleneck machines, developed in [16] for open lines, is extended to closed lines.
- Approach to determining the smallest (i.e., lean) values of N_0 and S , which ensures $PR_{cl} \approx PR_o$, is proposed.

To accomplish this, Section II below defines the model of the closed line under consideration and formulates the problems addressed. In Section III, a method for performance analysis is developed, monotonicity properties of PR_{cl} as a function of machine and buffer parameters are analyzed, and conditions are established, under which the closed nature of the line does not impede the open line performance. Improvability of closed lines with respect to S and N_0 is investigated in Section IV, and a method for bottleneck identification is developed in Section V. Section VI addresses the issue of (N_0, S) leanness. Finally, in Section VII, the conclusions are formulated. All proofs are given in the Appendix.

II. SYSTEM MODEL AND PROBLEM FORMULATION

A. Model

Consider a production line shown in Fig. 1. Assume that it operates according to the following assumptions.

- i) System consists of M machines m_i , $i = 1, \dots, M$, arranged serially, and $M - 1$ in-process buffers, b_i , $i = 1, \dots, M - 1$, separating each consecutive pair of machines.
- ii) Machines have identical cycle time τ . The time axis is slotted with the slot duration τ . The status of the machines (up or down) is determined at the beginning of each time slot.
- iii) Each in-process buffer b_i , $i = 1, \dots, M - 1$, is characterized by its capacity, N_i , $i = 1, \dots, M - 1$, where $1 \leq N_i < \infty$. The state of the buffer (i.e., the number of parts in it) is determined at the end of each time slot.
- iv) Machines obey the Bernoulli reliability model, i.e., m_i , $i = 1, \dots, M$, being neither blocked nor starved, produces a part during a time slot with probability p_i and fails to do so with probability $1 - p_i$. Parameter p_i is referred to as the *efficiency* of m_i .
- v) Machine m_i , $i = 2, \dots, M$, is starved during a time slot if buffer b_{i-1} is empty at the beginning of the time slot. Machine m_i , $i = 1, \dots, M - 1$, is blocked during a time slot if buffer b_i has N_i parts at the beginning of the time slot and machine m_{i+1} fails to take a part during the time slot.
- vi) Parts are transported within the system on carriers. The total number of carriers is S and the capacity of empty carrier buffer b_0 is $N_0 < \infty$.
- vii) Parts are placed on carriers at the input of machine m_1 . It is assumed that the parts are always available so that m_1 is not starved for parts but can be starved for carriers (when b_0 is empty).
- viii) Parts are removed from carriers at the output of machine m_M . It is assumed that m_M is not blocked by a subsequent operation but can be blocked by carriers (when b_0 is full and m_1 is either down or blocked).

Note that assumptions v) and viii) imply the *blocked before service* convention. This means, in particular, that a carrier with a part being processed by m_i is viewed as if it is already in b_i (or b_0 in the case of $i = M$). That is why the buffer capacity is defined in assumption iii) as being greater than or equal to 1.

Note also that assumptions i)–v) define an open line, corresponding to the closed line under consideration. Methods for analysis, continuous improvement, and design of such open lines are developed in [1].

B. Problems Addressed

In the framework of the above model, this paper addresses the problems listed below.

- 1) *Performance Analysis Problem*: Given the machine and in-process buffer parameters as well as N_0 and S , calculate the

production rate, and probability of blockages/starvations of the machines, i.e., evaluate

$$\begin{aligned} PR_{cl} &= PR_{cl}(p_1, \dots, p_M, N_1, \dots, N_{M-1}, N_0, S) \\ BL_i^{cl} &= BL_i^{cl}(p_1, \dots, p_M, N_1, \dots, N_{M-1}, N_0, S) \\ ST_i^{cl} &= ST_i^{cl}(p_1, \dots, p_M, N_1, \dots, N_{M-1}, N_0, S) \\ i &= 1, \dots, M. \end{aligned}$$

This is carried out in Section III along with the study of monotonicity properties of PR_{cl} with respect to p_i , N_i , and S . In addition, Section III discusses the issue of performance impediment defined as follows.

Definition 1: A pair (N_0, S) is *unimpeding* if

$$\begin{aligned} PR_{cl}(p_1, \dots, p_M, N_1, \dots, N_{M-1}, N_0, S) \\ = PR_o(p_1, \dots, p_M, N_1, \dots, N_{M-1}) \quad (2) \end{aligned}$$

otherwise, (N_0, S) is called *impeding*.

2) *Improvability Problem:* Improvability of open lines with Bernoulli machines has been analyzed in [1]. Here, we address improvability issues for closed lines. Specifically, assuming that the machine and in-process buffer parameters are fixed, i.e., p_i , $i = 1, \dots, M$, and N_i , $i = 1, \dots, M - 1$, are given, consider the function $PR_{cl}(N_0, S)$.

Definition 2: A closed line is

- S^+ -improvable if

$$PR_{cl}(N_0, S + 1) > PR_{cl}(N_0, S)$$

- S^- -improvable if

$$PR_{cl}(N_0, S - 1) > PR_{cl}(N_0, S)$$

- S -unimprovable if $PR_{cl}(N_0, S^*)$ cannot be increased for any other S , i.e.,

$$PR_{cl}(N_0, S^*) \geq PR_{cl}(N_0, S), \quad \forall S \neq S^*.$$

Note that the unimprovable value of S , i.e., S^* , may be nonunique, as illustrated in Fig. 2. Clearly, the unimprovable S^* is a function of N_0 , and is denoted throughout this paper as $S^* = S^*(N_0)$.

Definition 3: A closed line is N_0 -improvable if

$$PR_{cl}(N_0 + 1, S^*(N_0 + 1)) > PR_{cl}(N_0, S^*(N_0))$$

otherwise, it is *unimprovable* and the pair $(N_0^*, S^*(N_0^*))$ is called (N_0, S) -unimprovable.

Criteria for S^+ -, S^- -, and N_0 -improvability are given in Section IV.

3) *Bottleneck Identification Problem:* We use the definition of the bottleneck (BN) machine introduced in [16]:

Definition 4: Machine m_i , $i = 1, \dots, M$, is the *bottleneck* of a production line if

$$\frac{\partial PR}{\partial p_i} > \frac{\partial PR}{\partial p_j}, \quad \forall j \neq i \quad (3)$$

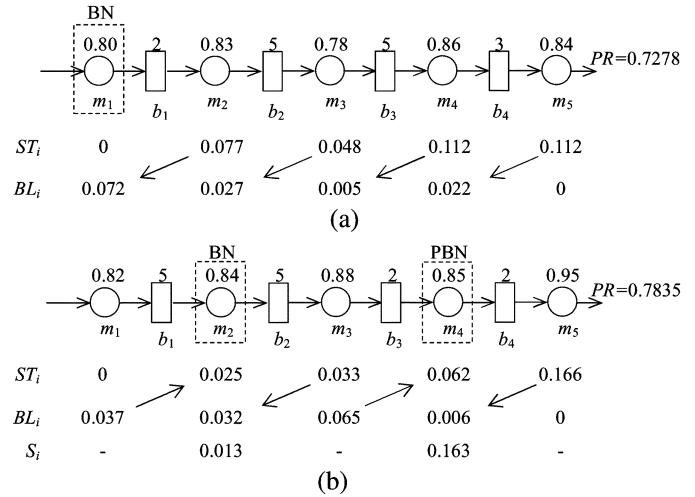


Fig. 3. Bottleneck identification in open serial lines. (a) Single bottleneck case. (b) Multiple bottlenecks case.

where PR is either PR_o or PR_{cl} —depending on which line is being considered.

Although (3) provides a definition of BNs, it can hardly be used for BN identification, since the partial derivatives involved in (3) can be neither measured on the factory floor nor calculated analytically. Therefore, an indirect method for verifying (3) and, thus, identifying the BN, is necessary. For the case of open lines, such a method has been developed in [16]. It is illustrated in Fig. 3 for two production lines with the machine and buffer parameters indicated therein. The method consists of calculating or measuring on the factory floor the probabilities of blockages, BL_i , and starvations, ST_i , of all machines in the system and assigning an arrow directed from m_i to m_{i+1} if $BL_i > ST_{i+1}$ and from m_{i+1} to m_i if $BL_i < ST_{i+1}$. It turns out [16] that if there is a single machine with no emanating arrows [see Fig. 3(a)], it is the BN in the sense of (3). If there are multiple machines with no emanating arrows [see Fig. 3(b)], the one with the largest severity is the primary bottleneck (PBN), where the severity of the bottleneck (i.e., each machine with no emanating arrows) is defined as

$$\begin{aligned} S_i &= |ST_{i+1} - BL_i| + |ST_i - BL_{i-1}|, \quad i = 2, \dots, M - 1, \\ S_1 &= |ST_2 - BL_1|, \\ S_M &= |ST_M - BL_{M-1}|. \end{aligned} \quad (4)$$

Since in the case of open lines $ST_1 = BL_M = 0$, there always exists at least one machine with no emanating arrows. In closed lines, however, this is not necessarily the case as illustrated in Fig. 4 where, in addition to the usual arrows, the arrow between m_1 and m_M is assigned according to the same rule: if $ST_1 > BL_M$, the arrow is directed to the left; if $ST_1 < BL_M$, it is directed to the right. In this situation, which machine is the BN? The answer to this question is provided in Section V.

4) *Problem of (N_0, S) Leanness:* Let $(N_0^*, S^*(N_0^*))$ denote an unimpeding pair. Introduce:

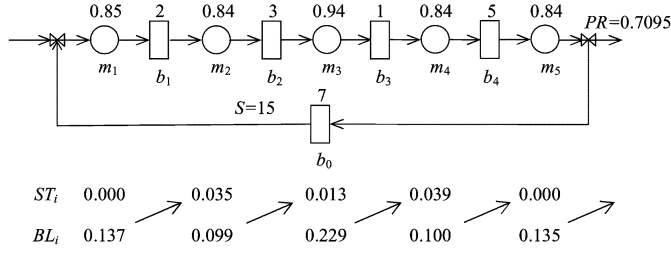


Fig. 4. Towards bottleneck identification in closed serial lines.

Definition 5: An unimpeding pair $(N_0^*, S^*(N_0^*))$ is *lean* if N_0^* and $S^*(N_0^*)$ are the smallest among all possible unimpeding pairs.

The issue of leanness is discussed in Section VI.

III. PERFORMANCE ANALYSIS, MONOTONICITY, AND UNIMPEDING CLOSED LINES

A. Two-Machine Lines

Theorem 1: The performance characteristics of a closed line defined by assumptions i)-viii) with $M = 2$ can be evaluated, as shown in (5)–(9) at the bottom of the page, where

$$\begin{aligned} PR_{cl} &= p_2[1 - Q^{cl}(p_1, p_2, N_1, N_0, S)] \\ &= p_1[1 - Q^{cl}(p_2, p_1, N_0, N_1, S)] \end{aligned} \quad (5)$$

$$BL_1^{cl} = \begin{cases} p_1 Q^{cl}(p_2, p_1, N_0, N_1, S), & \text{if } S > N_1 \text{ or } S = N_1 \geq N_0 \\ p_1(1 - p_2)Q^{cl}(p_2, p_1, N_0, N_1, S), & \text{if } S = N_1 < N_0 \\ 0, & \text{if } S < N_1 \end{cases} \quad (6)$$

$$ST_1^{cl} = \begin{cases} p_1 Q^{cl}(p_2, p_1, N_0, N_1, S), & \text{if } S \leq N_1 \\ 0, & \text{if } S > N_1 \end{cases} \quad (7)$$

$$BL_2^{cl} = \begin{cases} p_2 Q^{cl}(p_1, p_2, N_1, N_0, S), & \text{if } S > N_0 \text{ or } S = N_0 \geq N_1 \\ p_2(1 - p_1)Q^{cl}(p_1, p_2, N_1, N_0, S), & \text{if } S = N_0 < N_1 \\ 0, & \text{if } S < N_0 \end{cases} \quad (8)$$

$$ST_2^{cl} = \begin{cases} p_2 Q^{cl}(p_1, p_2, N_1, N_0, S), & \text{if } S \leq N_0 \\ 0, & \text{if } S > N_0 \end{cases} \quad (9)$$

$$\alpha(p_1, p_2) = \frac{p_1(1 - p_2)}{p_2(1 - p_1)} \quad (10)$$

$$Q^{cl}(p_1, p_2, N_1, N_0, S) = \begin{cases} Q_1^{cl}(p_1, p_2, N_1, N_0, S), & \text{if } S \leq \min(N_0, N_1) \\ Q_2^{cl}(p_1, p_2, N_1, N_0, S), & \text{if } \min N_i < S \leq \max N_i \\ Q_3^{cl}(p_1, p_2, N_1, N_0, S), & \text{if } S > \max(N_0, N_1) \end{cases} \quad (11)$$

$$Q_1^{cl}(p_1, p_2, N_1, N_0, S) = \begin{cases} \frac{1-p}{S+1-2p}, & \text{if } p_1 = p_2 = p \\ \frac{(1-p_1)[1-\alpha(p_1, p_2)]}{1 - \frac{p_1(1-p_1)}{p_2(1-p_2)}\alpha(p_1, p_2)^S}, & \text{if } p_1 \neq p_2 \end{cases} \quad (12)$$

$$Q_2^{cl}(p_1, p_2, N_1, N_0, S) = \begin{cases} \frac{1-p}{\min(N_1, N_0)+1-p}, & \text{if } p_1 = p_2 = p \\ \frac{(1-p_1)[1-\alpha(p_1, p_2)]}{1 - \frac{p_1}{p_2}\alpha(p_1, p_2)^{\min(N_1, N_0)}}, & \text{if } p_1 \neq p_2 \end{cases} \quad (13)$$

$$Q_3^{cl}(p_1, p_2, N_1, N_0, S) = \begin{cases} \frac{1-p}{N_1+N_0-S+1}, & \text{if } p_1 = p_2 = p \\ \frac{(1-p_1)[1-\alpha(p_1, p_2)]}{1-\alpha(p_1, p_2)^{N_1+N_0-S+1}}, & \text{if } p_1 \neq p_2 \end{cases} \quad (14)$$

$\alpha, Q^{cl}, Q_i^{cl}, i = 1, 2, 3$, are defined by (10)–(14) at the bottom of the page.

Proof: See the Appendix.

Corollary 1: Function $PR_{cl}(p_1, p_2, N_1, N_0, S)$ is

- strictly increasing in $p_i, i = 1, 2$;
- non-strictly increasing in N_1 and N_0 ;
- non-monotonic concave in S .

Proof: See the Appendix.

This corollary is of practical importance. First, it states that, similar to open lines, increasing p_i 's always leads to increased production rate in closed lines as well. Second, it states that, unlike open lines, increasing buffer capacity does not always lead to improved performance. Finally, it reaffirms the evidence of Fig. 2 that PR is non-monotonic concave in S .

Corollary 2: The pair (N_0, S) is unimpeding if and only if

$$N_1 < S \leq N_0. \quad (15)$$

Proof: See the Appendix.

Along with providing the unimpeding values of N_0 and S , this corollary has another important implication. It states that, in fact, unimpeding N_0 and S are independent of the machine efficiencies p_1 and p_2 : as long as (15) is observed, the closed

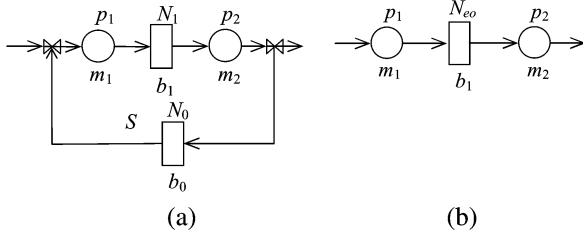


Fig. 5. Closed and equivalent open two-machine lines. (a) Closed line. (b) Equivalent open line.

nature of the line does not impede the open line behavior, no matter what p_1 and p_2 are. Thus, changing p_i 's cannot change an unimpeding pair (N_0, S) into an impeding one, and vice versa.

Concluding this subsection, we describe a property of asymptotic equivalence of closed and open lines. To accomplish this, consider the closed two-machine line of Fig. 5(a) and the open two-machine line of Fig. 5(b). Note that the efficiencies of the machines in these two lines are the same, while N_{eo} in Fig. 5(b) is not yet determined. Is it possible to select N_{eo} so that the production rates of these lines are the same or, at least, almost the same? Referring to the system of Fig. 5(b) as *equivalent open line*, the above question amounts to determining N_{eo} such that

$$PR_{cl}(p_1, p_2, N_1, N_0, S) \approx PR_{eo}(p_1, p_2, N_{eo}) \quad (16)$$

where PR_{eo} denotes the production rate of the equivalent open line. Let BL_1^{eo} and ST_2^{eo} denote the probabilities of blockage of m_1 and starvation of m_2 in the equivalent open line. Then:

Corollary 3: Assume that the machines are asymptotically reliable, i.e.,

$$p_i = 1 - \epsilon k_i, \quad i = 1, 2 \quad (17)$$

where $0 < \epsilon \ll 1$ and $k_i > 0$ is independent of ϵ . Then, if

$$N_{eo} = \begin{cases} S - 1, & \text{if } S \leq \min(N_1, N_0) \\ \min(N_1, N_0), & \text{if } \min N_i < S \leq \max N_i \\ N_1 + N_0 - S + 1, & \text{if } \max(N_1, N_0) < S \end{cases} \quad (18)$$

the performance characteristics of the closed two-machine line defined by assumptions i)–viii) and the equivalent open two-

machine line defined by i)–v) are related, as shown in (19)–(23) at the bottom of the page, where $O(\epsilon)$ and $O(\epsilon^2)$ denote terms of order of magnitude ϵ and ϵ^2 , respectively.

Proof: See the Appendix.

Thus, for closed lines with p_i 's close to 1, all the performance characteristics can be evaluated using the open lines expressions with the buffer capacity defined in (18). Note that this is an extension of the result obtained in [10].

B. $M > 2$ -Machine Lines

Similar to $M = 2$, the behavior of closed lines with $M > 2$ can be described by ergodic Markov chains. However, due to the complexity of their transition matrices, closed form expressions for the performance measures are all but impossible to derive. Therefore, we resort to a more restrictive statement.

Theorem 2: Assume that the production line defined by assumptions i)–viii) satisfies the following condition:

$$\sum_{i=1}^{M-1} N_i < S \leq N_0. \quad (24)$$

Then, the pair (N_0, S) is unimpeding and, therefore, all performance characteristics of this line coincide with those of the corresponding open line, i.e.,

$$\begin{aligned} PR_{cl}(p_1, \dots, p_M, N_1, \dots, N_0, S) &= PR_o(p_1, \dots, p_M, N_1, \dots, N_{M-1}), \\ BL_i^{cl}(p_1, \dots, p_M, N_1, \dots, N_0, S) &= BL_i^o(p_1, \dots, p_M, N_1, \dots, N_{M-1}), \quad i = 1, \dots, M, \\ ST_i^{cl}(p_1, \dots, p_M, N_1, \dots, N_0, S) &= ST_i^o(p_1, \dots, p_M, N_1, \dots, N_{M-1}), \quad i = 1, \dots, M. \end{aligned}$$

Proof: See the Appendix.

Theorem 2 implies, in particular, that under condition (24), the unimpeding pair (N_0, S) is independent of p_i 's. We show below that a similar property holds even without (24). Since we show this by simulations, the notion of unimpediment is modified to account for the fact that PR_{cl} and PR_o are determined with finite accuracy and, therefore, equality (2) may hold only approximately.

$$\begin{aligned} PR_{cl} &= PR_{eo}(p_1, p_2, N_{eo}) + O(\epsilon^2) \\ &= PR_{eo}(p_2, p_1, N_{eo}) + O(\epsilon^2) \end{aligned} \quad (19)$$

$$BL_1^{cl} = \begin{cases} BL_1^{eo}(p_1, p_2, N_{eo}) + O(\epsilon^2), & \text{if } S > N_1 \text{ or } S = N_1 \geq N_0 \\ O(\epsilon^2), & \text{if } S = N_1 < N_0 \\ 0, & \text{if } S < N_1 \end{cases} \quad (20)$$

$$ST_1^{cl} = \begin{cases} BL_1^{eo}(p_1, p_2, N_{eo}) + O(\epsilon^2), & \text{if } S \leq N_1 \\ 0, & \text{if } S > N_1 \end{cases} \quad (21)$$

$$BL_2^{cl} = \begin{cases} ST_2^{eo}(p_1, p_2, N_{eo}) + O(\epsilon^2), & \text{if } S > N_0 \text{ or } S = N_0 \geq N_1 \\ O(\epsilon^2), & \text{if } S = N_0 < N_1 \\ 0, & \text{if } S < N_0 \end{cases} \quad (22)$$

$$ST_2^{cl} = \begin{cases} ST_2^{eo}(p_1, p_2, N_{eo}) + O(\epsilon^2), & \text{if } S \leq N_0 \\ 0, & \text{if } S > N_0 \end{cases} \quad (23)$$

Definition 1': A pair (N_0, S) is *practically unimpeding* if

$$\frac{|PR_{cl}^{mes} - PR_o^{mes}|}{PR_o^{mes}} \leq \delta \ll 1$$

where

$$PR_{cl}^{mes} = PR_{cl}^{mes}(p_1, \dots, p_M, N_1, \dots, N_{M-1}, N_0, S)$$

$$PR_o^{mes} = PR_o^{mes}(p_1, \dots, p_M, N_1, \dots, N_{M-1})$$

are estimates of PR_{cl} and PR_o , respectively, obtained from measurements and δ can be selected by the practitioner according to the accuracy requirements.

Numerical Fact 1: In closed lines defined by assumptions i)–viii), a practically unimpeding pair (N_0, S) remains practically unimpeding for all values of p_i , $i = 1, \dots, M$, as long as N_0, N_1, \dots, N_{M-1} remain the same.

As it was mentioned above, the justification of Numerical Fact 1 is based on simulations. Since an analogous approach is used elsewhere in this paper, we describe it below as a standard procedure.

Numerical Simulation Procedure 1:

- A C++ code to simulate the production system defined by assumptions i)–viii) is constructed.
- The initial status of each machine is selected up with probability p_i and down with probability $1 - p_i$, $i = 1, \dots, M$, respectively.
- Carriers are initially placed in the empty carrier buffer, with the excess carriers (if any) placed randomly and equiprobably in in-process buffers.
- For each line under consideration, 20 runs of the simulation code are carried out.
- In each run, the first 20 000 time slots are used as a warm-up period and the subsequent 200 000 time slots are used to statistically evaluate the performance measures of interest.
- This results in PR_{cl}^{mes} , ST_i^{mes} and BL_i^{mes} , which provide estimates of PR , ST_i and BL_i with 95% confidence intervals 0.001 for PR and 0.002 for ST_i and BL_i .

Although for open lines, all performance measures could be evaluated analytically, for the sake of consistency, we used the above simulation approach to evaluate PR_o^{mes} as well.

Justification of Numerical Fact 1: We constructed 30 000 closed lines by selecting M , p_i 's, and N_i 's randomly and equiprobably from the following sets, respectively

$$M \in \{3, 5, 10\} \quad (25)$$

$$p_i \in [0.7, 0.95], \quad i = 1, \dots, M \quad (26)$$

$$N_i \in \{1, 2, 3, 4, 5\}, \quad i = 1, \dots, M - 1 \quad (27)$$

$$N_0 \in \left\{ 3, 4, \dots, \sum_{i=1}^{M-1} N_i - 1 \right\}. \quad (28)$$

Note that condition (24) does not take place on set (28). In addition, $\delta = 0.01$ is used.

For each of these lines, using Numerical Simulation Procedure 1, we evaluated PR_{cl}^{mes} and PR_o^{mes} and selected randomly and equiprobably a pair $(N_0^*, S^*(N_0^*))$, which was practically unimpeding in the sense of Definition 1'. To verify whether this

pair remains practically unimpeding with other values of p_i 's, for each of the 30 000 lines mentioned above, we constructed 10 more lines with N_i , $i = 0, 1, \dots, M - 1$, being the same but with new p_i 's selected randomly and equiprobably from set (26). Again, using Numerical Simulation Procedure 1, we evaluated PR_{cl}^{mes} and PR_o^{mes} for each of the new lines, thus constructed, and verified whether Definition 1' holds. As a result, we determined that $(N_0^*, S^*(N_0^*))$ remains unimpeding in 99.52% of the cases analyzed. Thus, we conclude that Numerical Fact 1 holds.

In conclusion of this subsection, guided by Corollary 1, we formulate established:

Numerical Fact 2: Function $PR_{cl}(p_1, \dots, p_M, N_1, \dots, N_{M-1}, N_0, S)$ is

- strictly increasing in p_i , $i = 1, \dots, M$;
- nonstrictly increasing in N_i , $i = 0, \dots, M - 1$;
- nonmonotonic concave in S .

Justification of Numerical Fact 2: This justification has been carried out using Numerical Simulation Procedure 1. In all cases analyzed, no counter-examples have been found.

IV. IMPROVABILITY

The definitions of S - and N_0 -improvability are given in Section II-B2. Here we provide methods for identifying whether a line is improvable in the appropriate sense or not. Throughout, we denote as S_{\min}^* and S_{\max}^* the smallest and largest unimprovable S . Clearly, in some systems $S_{\min}^* = S_{\max}^*$ (see Fig. 2).

A. Two-Machine Lines

Theorem 3: For $S \notin [S_{\min}^*, S_{\max}^*]$, a closed line defined by assumptions i)–viii) with $M = 2$ is

- S^+ -improvable if

$$\sum_{i=1}^2 ST_i > \sum_{i=1}^2 BL_i.$$

- S^- -improvable if

$$\sum_{i=1}^2 ST_i < \sum_{i=1}^2 BL_i.$$

Proof: See the Appendix.

For $S \in [S_{\min}^*, S_{\max}^*]$, increasing or decreasing S leads to a limit cycle, i.e., “oscillations” between S and $S - 1$ or $S + 1$. In this case, the best S (i.e., the one resulting in the largest PR) must be selected from the limit cycle.

It is convenient to introduce the notation

$$I = \sum_{i=1}^M (ST_i - BL_i) \quad (29)$$

and refer to I as the S -improvability indicator. Thus, positive (respectively, negative) I 's imply S^+ - (respectively, S^- -) improvability.

In addition to its direct value as a tool for S -improvability identification, the utility of the above theorem (and the subsequent statement for $M > 2$) is in the fact that S -improvability

can be identified without knowing the machine and buffer parameters but just by measuring the frequency of blockages and starvations of the machines during normal system operation.

Finally, we formulate the following.

Corollary 4: If a system is S -unimprovable and ST_1 and/or BL_2 are nonzero, then the system is N_0 -improvable.

Proof: Follows directly from Theorem 3 and the definition of N_0 -improvability.

B. M -Machine Lines

Due to complexities of the Markov chains, which describe closed lines with more than two machines, an extension of Theorem 3 for the case of $M > 2$ is all but impossible to derive analytically. However, based on simulations, we conclude that it takes place for any M . Specifically, we have the following.

Numerical Fact 3: For $S \notin [S_{\min}^*, S_{\max}^*]$, a closed line defined by assumptions i)-viii) with $M > 2$ is

- S^+ -improvable if

$$\sum_{i=1}^M ST_i > \sum_{i=1}^M BL_i.$$

- S^- -improvable if

$$\sum_{i=1}^M ST_i < \sum_{i=1}^M BL_i.$$

Thus, I of (29) is still the indicator of improvability: S^+ if I is positive and S^- if I is negative.

Justification of Numerical Fact 3: We constructed 300 000 closed lines by selecting M , p_i 's, and N_i 's randomly and equiprobably from the following sets, respectively

$$M \in \{3, 5, 10\}, \quad (30)$$

$$p_i \in [0.7, 0.95], \quad i = 1, \dots, M \quad (31)$$

$$N_i \in \{1, 2, 3, 4, 5\}, \quad i = 1, \dots, M - 1 \quad (32)$$

$$N_0 \in \{1, 2, \dots, 15\}. \quad (33)$$

For each of these lines, using Numerical Simulation Procedure 1, we evaluated $PR_{cl}^{mes}(S)$ for all $S \in \{1, 2, \dots, \sum_{i=0}^{M-1} N_i - 1\}$ and determined S_{opt} for which $PR_{cl}^{mes}(S)$ is maximized. Also, for each of these lines, using Numerical Simulation Procedure 1 and Numerical Fact 3, we obtained S_{unimp} , i.e., the value of S at which the improvability indicator I changes its sign. Then, we compared the values of $PR_{cl}^{mes}(S_{opt})$ and $PR_{cl}^{mes}(S_{unimp})$. As a result, we determined that the two production rates are within 1% of each other in 99.51% of the cases analyzed. Thus we conclude that Numerical Fact 3 indeed defines the conditions of S -improvability.

Based on this, we formulate:

Continuous Improvement Procedure With Respect to S :

- 1) Evaluate ST_i and BL_i for all machines in the system.
- 2) Calculate the S -improvability indicator $I = \sum_{i=1}^M (ST_i - BL_i)$.
- 3) If $I > 0$, increase S by one; if $I < 0$, decrease S by one.
- 4) Return to 1) and continue until a limit cycle is reached.
- 5) Select the S from the limit cycle, which gives the largest PR ; this S is unimprovable and is denoted as $S^*(N_0)$.

Clearly, if for the above $S^*(N_0)$

$$PR_{cl}(N_0, S^*(N_0)) < PR_{cl}(N_0 + 1, S^*(N_0 + 1))$$

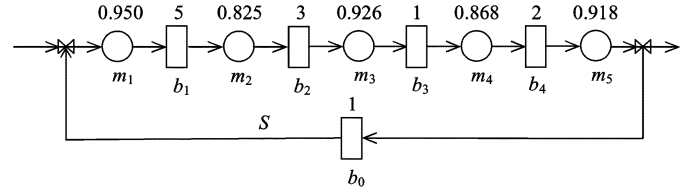


Fig. 6. Production line for Example 1.

TABLE I
CONTINUOUS IMPROVEMENT PROCEDURE WITH RESPECT TO S
FOR EXAMPLE 1 (STARTING FROM $S = 2$)

	S	I	PR
1	2	2.3126	0.4189
2	3	1.4071	0.5877
3	4	0.7802	0.7006
4	5	0.4940	0.7447
5	6	0.3302	0.7623
6	7	0.2572	0.7664
7	8	0.1922	0.7673
8	9	0.0841	0.7673
9	10	-0.0577	0.7674
10	9	0.0841	0.7673

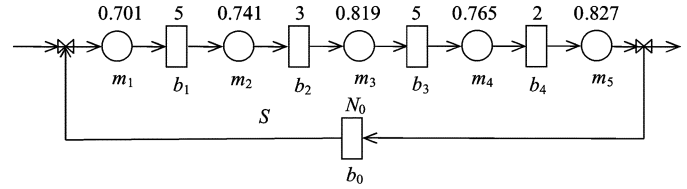


Fig. 7. Production line for Example 2.

the pair $(N_0, S^*(N_0))$ is impeding and, therefore, is improvable with respect to N_0 . This improvement can be carried out using the following.

Continuous Improvement Procedure With Respect to N_0 :

- 1) For a given N_0 and $N_0 + 1$, carry out the Continuous Improvement Procedure with respect to S and determine $S^*(N_0)$ and $S^*(N_0 + 1)$.
- 2) If $|PR_{cl}(N_0, S^*(N_0)) - PR_{cl}(N_0 + 1, S^*(N_0 + 1))| > \delta PR_{cl}(N_0, S^*(N_0))$, $\delta \ll 1$, increase N_0 by one, return to 1).
- 3) If $|PR_{cl}(N_0, S^*(N_0)) - PR_{cl}(N_0 + 1, S^*(N_0 + 1))| \leq \delta PR_{cl}(N_0, S^*(N_0))$, $\delta \ll 1$, the system is unimprovable with respect to N_0 ; this N_0 and the resulting S is an unimpeding pair and is denoted as $(N_0^*, S^*(N_0^*))$.

Below, two examples illustrating these procedures are given. In the first example, the system of Fig. 6 is considered and the Continuous Improvement Procedure with respect to S is carried out starting from $S = 2$ and $S = 21$. The results are given in Table I and II, respectively. In both cases, the unimprovable number of carriers is 10.

In the second example, the Continuous Improvement Procedure with respect to N_0 (for $\delta = 0.01$) is applied to the system of Fig. 7. As a result, an unimprovable pair $(N_0^*, S^*(N_0^*))$ is obtained with $N_0^* = 5$ and $S^*(N_0^*) = 11$, as shown in Table III.

C. Comparisons

Reference [12] offers an interesting formula for selecting S in closed lines with machines having random processing time and with blocked after service (BAS) convention (which implies that

TABLE II
 CONTINUOUS IMPROVEMENT PROCEDURE WITH RESPECT TO S
 FOR EXAMPLE 1 (STARTING FROM $S = 21$)

Step #	S	I	PR
0	21	-0.4005	0.7596
1	20	-0.1886	0.7665
2	19	-0.1111	0.7671
3	18	-0.0936	0.7672
4	17	-0.0927	0.7673
5	16	-0.0940	0.7673
6	15	-0.0916	0.7672
7	14	-0.0934	0.7672
8	13	-0.0917	0.7666
9	12	-0.0921	0.7672
10	11	-0.0888	0.7672
11	10	-0.0577	0.7674
12	9	0.0841	0.7673
13	10	-0.0577	0.7674

 TABLE III
 CONTINUOUS IMPROVEMENT PROCEDURE WITH RESPECT TO N_0
 FOR EXAMPLE 2 ($PR_o = 0.6665$)

Step #	N_0	S	I	PR
0	2	5	0.7551	0.5823
1	2	6	0.5114	0.6116
2	2	7	0.3380	0.6287
3	2	8	0.2045	0.6386
4	2	9	0.0693	0.6425
5	2	10	-0.1017	0.6439
6	3	10	0.0367	0.6531
7	3	11	-0.1200	0.6530
8	4	10	0.1345	0.6580
9	4	11	0.0094	0.6597
10	4	12	-0.1337	0.6575
11	5	11	0.1055	0.6627
12	5	12	-0.0081	0.6618

even if the downstream buffer is full, a machine can process a part). This formula is

$$\widehat{S} = M + \left\lceil \frac{\sum_{i=0}^{M-1} N_i^{\text{BAS}}}{2} \right\rceil \quad (34)$$

where N_i^{BAS} , $i = 0, 1, \dots, M-1$, is the i th buffer capacity under the BAS convention and $\lceil x \rceil$ denotes the smallest integer not less than x . The blocked before service (BBS) convention, used in this paper, implies that the machine itself is a unit of buffer capacity; therefore

$$N_i^{\text{BAS}} = N_i - 1 \quad (35)$$

where N_i is the i th buffer capacity under BBS convention. Thus, formula (34) for systems under the BBS convention becomes

$$\widehat{S} = M + \left\lceil \frac{\sum_{i=0}^{M-1} (N_i - 1)}{2} \right\rceil = \left\lceil \frac{M + \sum_{i=0}^{M-1} N_i}{2} \right\rceil. \quad (36)$$

To investigate the relationship between S^* obtained by the Continuous Improvement Procedure with respect to S and \widehat{S} provided by expression (36), we use the examples of the previous subsection. The results are as follows: In the first example,

 TABLE IV
 COMPARISON OF THE CONTINUOUS IMPROVEMENT PROCEDURE
 WITH RESPECT TO S AND (36)

N_0	S_{opt}	$PR(S_{opt})$	S^*	$PR(S^*)$	\widehat{S}	$PR(\widehat{S})$
2	10	0.6439	10	0.6439	11	0.6438
3	10	0.6531	10	0.6531	12	0.6526
4	12	0.6600	11	0.6597	12	0.6600
5	12	0.6630	11	0.6627	13	0.6629

$S^* = 10$, while $\widehat{S} = 14$. This leads to $PR(S^*) = 0.7674$ and $PR(\widehat{S}) = 0.7672$. For the second example, the results are summarized in Table IV. As one can see, both approaches lead to similar outcomes with S^* being somewhat smaller than \widehat{S} .

V. BOTTLENECK IDENTIFICATION

A. Two-Machine Lines

Theorem 4: In closed lines defined by assumptions i)–viii) with $M = 2$, machine m_1 (respectively, machine m_2) is the bottleneck (BN) if and only if

$$ST_1 + BL_1 < ST_2 + BL_2, \quad (\text{respectively, } ST_1 + BL_1 > ST_2 + BL_2). \quad (37)$$

Proof: Follows the proof of the BN theorem obtained in [16] for open lines.

This theorem can be interpreted as follows: Let us refer to $ST_1 + BL_1$ as the *virtual blockage* of m_1 and to $ST_2 + BL_2$ as the *virtual starvation* of m_2 , i.e.,

$$\begin{aligned} BL_{1,v} &= ST_1 + BL_1 \\ ST_{2,v} &= ST_2 + BL_2 \end{aligned}$$

and assume that the virtual starvation of m_1 and virtual blockage of m_2 are 0, i.e.,

$$ST_{1,v} = BL_{2,v} = 0. \quad (38)$$

Assign arrows from between m_1 and m_2 according to the same rule as in the case of open lines but using virtual blockages and starvations of m_1 and m_2 . Then, according to Theorem 4, the machine with no emanating arrows is the BN of the closed line. This is illustrated in Fig. 8. Thus, using virtual, rather than real, blockages and starvations allows us to extend the open line BN identification technique to closed ones. As shown below, this can be done for $M > 2$ as well.

B. $M > 2$ -Machine Lines

Consider an $M > 2$ -machine closed line and assume that ST_i and BL_i , $i = 1, \dots, M$, are identified during normal system operation. Similar to the case of $M = 2$, introduce the virtual blockages and starvations of the machines as follows:

$$\begin{aligned} BL_{1,v} &:= ST_1 + BL_1, \\ ST_{1,v} &:= 0, \\ BL_{i,v} &:= BL_i, \quad i = 2, \dots, M-1, \\ ST_{i,v} &:= ST_i, \quad i = 2, \dots, M-1, \\ BL_{M,v} &:= 0, \\ ST_{M,v} &:= ST_M + BL_M. \end{aligned}$$

Using $ST_{i,v}$ and $BL_{i,v}$, assign arrows between m_i and m_{i+1} according to the same rule as in the open lines, i.e., an arrow is directed from m_i to m_{i+1} if $BL_{i,v} > ST_{i+1,v}$ and from m_{i+1}

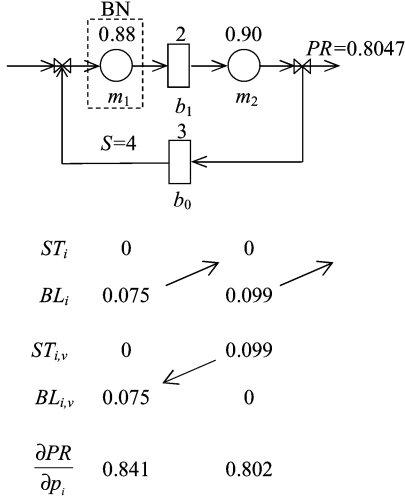


Fig. 8. BN identification in two-machine closed lines.

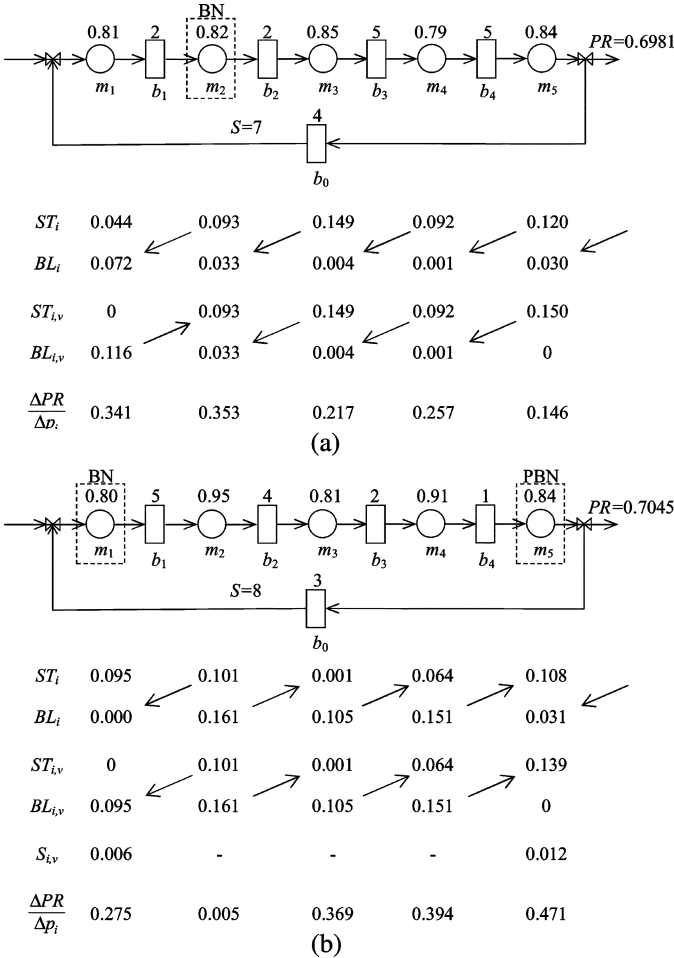


Fig. 9. Bottleneck identification in five-machine closed lines. (a) Single bottleneck case. (b) Multiple bottlenecks case.

to m_i if $BL_{i,v} < ST_{i+1,v}$. Since $ST_{1,v} = BL_{M,v} = 0$, there is at least one machine with no emanating arrow (see Fig. 9).

Numerical Fact 4: If there is a single machine with no emanating arrows, it is the BN of the closed line. If there are multiple machines with no emanating arrows, the one with the largest virtual severity is the PBN, where the virtual severity is defined as

$$S_{i,v} := \max(|ST_{i+1,v} - BL_{i,v}|, |ST_i - BL_{i-1,v}|),$$

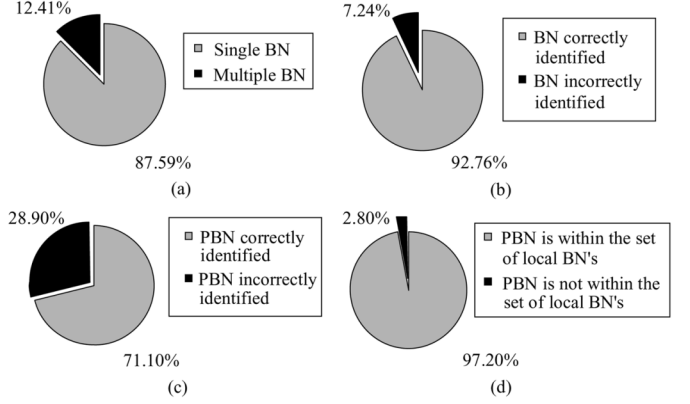


Fig. 10. Accuracy of Numerical Fact 4. (a) Single versus Multiple BN. (b) Accuracy of BN identification in single BN case. (c) Accuracy of PBN identification. (d) PBN is within the set of local BN's.

$$i = 2, \dots, M-1,$$

$$S_{1,v} := \max(|ST_{2,v} - BL_{1,v}|, |ST_{1,v} - BL_{M,v}|),$$

$$S_{M,v} := \max(|ST_{M,v} - BL_{M-1,v}|, |ST_{1,v} - BL_{M,v}|). \quad (39)$$

Justification of Numerical Fact 4: This justification has been carried out as follows: A total of 1 000 000 closed lines have been generated with parameters selected randomly and equiprobably from sets (30)–(33) and $S \in \{M, M+1, \dots, \sum_{i=0}^{M-1} N_i\}$. Each of these lines has been analyzed using Numerical Simulation Procedure 1. Specifically, the probabilities of blockages and starvations of all machines have been estimated and, in addition, partial derivatives of the production rate with respect to p_i 's have been evaluated. The probabilities of blockages and starvations have been used to identify the BN using Numerical Fact 4, and the partial derivatives have been used to identify the BN using Definition 4. If the BN identified by both method were the same, we concluded that Numerical Fact 4 holds for the system at hand; otherwise, we concluded that it does not.

The results obtained using this approach are summarized in Fig. 10. Among the 1 000 000 lines analyzed, 87.59% had a single machine with no emanating arrows, and the BN machine was identified by Numerical Fact 4 correctly in 92.76% of these cases. For the 12.41% of the systems with more than one machine having no emanating arrows, Numerical Fact 4 identified correctly the PBN in 71.1% of the cases, while the PBN was indeed in the set of local BN's in 97.20% of the cases. These results are similar to those obtained in [16] for BN identification in open lines. Thus, we conclude that Numerical Fact 4 provides a sufficiently accurate tool for bottleneck identification in closed lines.

VI. LEANNESS

In this section, we discuss the selection of the smallest N_0 , i.e., N_0^{lean} , and the corresponding smallest S , i.e., $S_{\text{lean}}(N_0^{\text{lean}})$, which result in $PR_{cl} = PR_o$. Such a pair, as defined in Section II-B4, is called lean.

For two-machine lines, the lean pair (N_0, S) can be obtained immediately from Corollary 2

$$S_{\text{lean}}(N_0^{\text{lean}}) = N_0^{\text{lean}} = N_1 + 1. \quad (40)$$

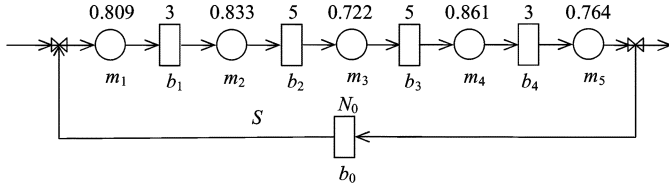

 Fig. 11. Example of lean (N_0, S) design procedure.

 TABLE V
 EXAMPLE OF LEAN (N_0, S) DESIGN PROCEDURE

Step #	N_0	S^*	$PR(S^*)$
0	13	13	0.7001
1	12	13	0.7001
2	11	13	0.7000
3	10	13	0.7000
4	9	13	0.7001
5	8	13	0.7000
6	7	12	0.6994
7	6	12	0.6994
8	5	12	0.6989
10	4	11	0.6982
11	3	11	0.6910

For $M > 2$, the pair $(N_0^{\text{lean}}, S_{\text{lean}})$ can be evaluated approximately using the following.

Lean (N_0, S) Design Procedure:

- 1) Using the Continuous Improvement Procedure with respect to N_0 and S , determine an unimpeding pair $(N_0^*, S^*(N_0^*))$.
- 2) Decrease N_0^* by 1 and determine $S^*(N_0^* - 1)$.
- 3) If $|PR_{cl}(N_0^* - 1, S^*(N_0^* - 1)) - PR_{cl}(N_0^*, S^*(N_0^*))| < \delta PR(N_0^*, S^*(N_0^*))$, $\delta \ll 1$, return to step 2).
- 4) If $|PR_{cl}(N_0^* - 1, S^*(N_0^* - 1)) - PR_{cl}(N_0^*, S^*(N_0^*))| > \delta PR(N_0^*, S^*(N_0^*))$, $\delta \ll 1$, then $N_0^* = N_0^{\text{lean}}$.

As an example, this procedure is applied to the system of Fig. 11 for $\delta = 0.01$, and the results are given in Table V. Clearly, it leads to the reduction of N_0 from 13 to 4 and S^* from 13 to 11, practically without losses in the production rate.

As it follows from (40), a lean pair $(N_0^{\text{lean}}, S_{\text{lean}})$ is independent of machine efficiencies in two-machine lines. It is possible to show by contradiction that this property holds for $M > 2$ -machine lines as well.

Indeed, assume that $(N_0^{\text{lean}}, S_{\text{lean}})$ is the lean pair for a system with one set of machines efficiencies and $(N_0^{\text{lean}} - 1, S_{\text{lean}})$ is the lean pair for the same system but with another set of machines efficiencies. Therefore, $(N_0^{\text{lean}} - 1, S_{\text{lean}})$ must be impeding for the first set of p_i 's. However, by Numerical Fact 1, practically unimpeding pairs are independent of machine efficiencies. Thus, the assumption is not true, and the conclusion is that the lean pair $(N_0^{\text{lean}}, S_{\text{lean}})$ is independent of p_i , $i = 1, \dots, M$.

VII. CONCLUSIONS

The performance of closed production lines can be impeded, in comparison with corresponding open lines, if the number of carriers, S , and the capacity of the empty carrier buffer, N_0 , are not selected correctly. This paper provides tools for determining if this impediment takes place and, if it does, offers methods for (N_0, S) improvement. Specifically, the following results are obtained.

- Method for calculating performance measures in two-machine closed Bernoulli lines is derived, and a more restrictive result for longer lines is obtained.

- Criterion of improvability with respect to the number of carriers is established. Specifically, if $\sum_{i=1}^M ST_i > \sum_{i=1}^M BL_i$, (respectively, $\sum_{i=1}^M ST_i < \sum_{i=1}^M BL_i$), the system is S^+ - (respectively, S^- -) improvability, i.e., PR can be increased by adding (respectively, removing) a carrier.
- Criterion of improvability with respect to the capacity of the empty carrier buffer is derived, and a corresponding continuous improvement procedure is proposed.
- Method for identifying bottleneck machines in closed lines is suggested. Specifically, it is shown that bottlenecks in closed lines can be identified based on the same procedure as that for open lines but using the so-called virtual, rather than real, probabilities of blockages and starvations.
- Procedure for calculating the lean empty carrier buffer capacity and the lean number of carriers is proposed.

As topics for future work, the following can be mentioned.

- Extensions of the results obtained to other than Bernoulli reliability models (e.g., exponential, Weibull, and, perhaps, general).
- Extensions to assembly systems.
- Extensions to systems with machines having different cycles times (i.e., the asynchronous case).
- Extensions to systems with rework and with multiple closed loops.
- Analysis of transients in closed serial lines and assembly systems.

APPENDIX

Proof of Theorem 1: Under assumptions i)–viii), the system under consideration is described by an ergodic Markov chain with states being the probability of occupancy of buffer b_1 (since, given the probability of occupancy of b_1 , the probability of occupancy of b_0 can be immediately calculated). Let P_j be the stationary probability that b_1 contains j parts, i.e.,

$$P_j = P\{h_1 = j\}, \quad j = 0, 1, \dots, N_1 \quad (\text{A.41})$$

where h_1 is the number of parts in b_1 . Then, the performance analysis of the system at hand amounts to evaluating P_j 's, and then evaluating PR , ST_i and BL_i , $i = 1, 2$. It turns out that it is convenient to calculate P_j separately for three cases of relationships among S , N_0 , and N_1 . This is carried out below.

Case 1: $S < \min(N_1, N_0)$. The balance equations for this case are

$$\begin{aligned} P_0 &= (1 - p_1)P_0 + (1 - p_1)p_2P_1 \\ P_1 &= p_1P_0 + [p_1p_2 + (1 - p_1)(1 - p_2)]P_1 \\ &\quad + (1 - p_1)p_2P_2 \\ P_i &= p_1(1 - p_2)P_{i-1} \\ &\quad + [p_1p_2 + (1 - p_1)(1 - p_2)]P_i \\ &\quad + (1 - p_1)p_2P_{i+1}, \quad i = 2, \dots, S - 1 \\ P_S &= p_1(1 - p_2)P_{S-1} + (1 - p_2)P_S, \\ P_j &= 0, \quad j = S + 1, \dots, N_1 \end{aligned}$$

$$\sum_{i=0}^{N_1} P_i = 1.$$

Their solution is

$$\begin{aligned} P_i &= \frac{\alpha^i}{1-p_2} P_0, \quad i = 1, \dots, S-1 \\ P_S &= \frac{\alpha^S}{1-p_2} (1-p_1) P_0 \\ P_j &= 0, \quad j = S+1, \dots, N_1 \end{aligned} \quad (\text{A.42})$$

where

$$\alpha(p_1, p_2) = \frac{p_1(1-p_2)}{(1-p_1)p_2} \quad (\text{A.43})$$

$$\begin{aligned} P_0 &=: Q_1^{cl}(p_1, p_2, N_1, N_0, S) \\ &= \begin{cases} \frac{1-p}{S+1-2p}, & \text{if } p_1 = p_2 = p \\ \frac{(1-p_1)[1-\alpha(p_1, p_2)]}{1-\frac{p_1(1-p_1)}{p_2(1-p_2)}\alpha(p_1, p_2)^S}, & \text{if } p_1 \neq p_2 \end{cases} \end{aligned} \quad (\text{A.44})$$

Case 2a: $N_1 < S \leq N_0$. In this case, m_1 is never starved and m_2 is never blocked. In other words, the closed loop does not impede the open loop performance. Thus, the stationary probability mass function is the same as the corresponding open line (see [1]). Therefore, as it follows from [1], for $N_1 < S \leq N_0$

$$\begin{aligned} P_0 &=: Q_2^{cl}(p_1, p_2, N_1, N_0, S) \\ &= \begin{cases} \frac{1-p}{\min(N_1, N_0)+1-p}, & \text{if } p_1 = p_2 = p \\ \frac{(1-p_1)[1-\alpha(p_1, p_2)]}{1-\frac{p_1(1-p_1)}{p_2(1-p_2)}\alpha(p_1, p_2)^{\min(N_1, N_0)}}, & \text{if } p_1 \neq p_2 \end{cases} \end{aligned} \quad (\text{A.45})$$

Case 2b: $N_0 < S \leq N_1$. Here, in the reversed flow scenario, the first machine, m_2 , is never starved and the second machine, m_1 , is never blocked. Thus, the line again is equivalent to an open line with the same machines but with the in-process buffer of capacity N_0 . Therefore, in this case

$$P_0 = Q_2^{cl}(p_2, p_1, N_0, N_1, S). \quad (\text{A.46})$$

Case 3: $S > \max(N_1, N_0)$. The balance equations in this case are

$$\begin{aligned} P_j &= 0, \quad j = 0, 1, \dots, S - N_0 - 1, \\ P_{S-N_0} &= [(1-p_1) + p_1 p_2] P_{S-N_0} + p_2(1-p_1) P_{S-N_0+1}, \\ P_i &= P_{i-1} p_1(1-p_2) + [p_1 p_2 + (1-p_1)(1-p_2)] P_i \\ &\quad + (1-p_1) p_2 P_{i+1}, \quad i = S - N_0 + 1, \dots, N_1 - 1, \\ P_{N_1} &= P_{N_1-1} p_1(1-p_2) + (p_1 p_2 + 1 - p_2) P_{N_1}, \\ \sum_{i=0}^{N_1} P_i &= 1. \end{aligned}$$

Their solution is

$$P_i = \alpha(p_1, p_2)^{i-(S-N_0)} P_{S-N_0}, \quad i = S - N_0, \dots, N_1 \quad (\text{A.47})$$

where $\alpha(p_1, p_2)$ is given in (A.43) and

$$P_{S-N_0} = \frac{1}{\sum_{i=0}^{N_1+N_0-S} \alpha(p_1, p_2)^i}. \quad (\text{A.48})$$

Given the above, the probability that b_0 is full and m_1 is down can be expressed as

$$\begin{aligned} Q_3^{cl}(p_1, p_2, N_1, N_0, S) \\ = \begin{cases} \frac{1-p}{N_1+N_0-S+1}, & \text{if } p_1 = p_2 = p \\ \frac{(1-p_1)[1-\alpha(p_1, p_2)]}{1-\alpha(p_1, p_2)^{N_1+N_0-S+1}}, & \text{if } p_1 \neq p_2 \end{cases} \end{aligned} \quad (\text{A.49})$$

Using the three probability mass functions derived above and following the same arguments as in [1], we obtain the expressions for the performance measures (5)–(9) given in Theorem 1. ■

Proof of Corollary 1: In this proof, we again consider three cases.

Case 1: $S < \min(N_1, N_0)$.

For $p_1 \neq p_2$, function $Q_1^{cl}(p_1, p_2, N_1, N_0, S)$ given in (A.44) can be rewritten as

$$\begin{aligned} Q_1^{cl}(p_1, p_2, N_1, N_0, S) \\ = \frac{(1-p_1)}{1 + \alpha + \dots + \alpha^{S-3} + \left[1 + \frac{p_1(1-p_2)(p_1+p_2)}{p_2^2}\right] \alpha^{S-2}}. \end{aligned}$$

Clearly, it is strictly decreasing in p_1 and S . Similarly, $Q_1^{cl}(p_2, p_1, N_0, N_1, S)$ is strictly decreasing in p_2 and S . Thus

$$\begin{aligned} PR_{cl} &= p_2 [1 - Q_1^{cl}(p_1, p_2, N_1, N_0, S)] \\ &= p_1 [1 - Q_1^{cl}(p_2, p_1, N_0, N_1, S)] \end{aligned}$$

is strictly increasing in p_1, p_2 , and S , and is independent of, i.e., constant in, N_1 and N_0 .

For $p_1 = p_2 = p$, it is easy to show that

$$PR_{cl} = \frac{p(S-p)}{S+1-2p} = p - \frac{p(1-p)}{S+1-2p}, \quad S \geq 2$$

which again implies that PR_{cl} is strictly increasing in p and S , and is independent of, i.e., constant in, N_1 and N_0 .

Case 2: $\min(N_1, N_0) < S \leq \max(N_1, N_0)$.

In this situation, the closed line is exactly equivalent to an open line. Thus, PR_{cl} is strictly increasing p_1 and p_2 , monotonically increasing in N_1 and N_0 and independent of, i.e., constant in S .

Case 3: $S > \max(N_1, N_0)$.

For $p_1 \neq p_2$

$$Q_3^{cl}(p_1, p_2, N_1, N_0, S) = \frac{(1-p_1)}{1 + \alpha + \dots + \alpha^{N_1+N_0-S}}$$

which implies that this function is strictly decreasing in p_1 , strictly decreasing in N_1 and N_0 , and strictly increasing in S . Therefore

$$\begin{aligned} PR_{cl} &= p_2 [1 - Q_3^{cl}(p_1, p_2, N_1, N_0, S)] \\ &= p_1 [1 - Q_3^{cl}(p_2, p_1, N_0, N_1, S)] \end{aligned}$$

is strictly increasing in p_1 and p_2 , strictly decreasing in S , and strictly increasing in N_1 and N_0 .

For $p_1 = p_2 = p$

$$PR_{cl} = \frac{p(N_1 + N_0 - S - p)}{N_1 + N_0 - S + 1}, \quad S \geq 2$$

and the same conclusions hold.

Thus, PR_{cl} is strictly increasing in p_1 and p_2 , nonstrictly increasing in N_1 and N_0 , and nonmonotonic concave in S . ■

Proof of Corollary 2: It has been shown in [1] that

$$PR_o(p_1, p_2, N_1) = p_2[1 - Q_o(p_1, p_2, N_1)] \quad (\text{A.50})$$

where

$$Q_o(p_1, p_2, N_1) = \begin{cases} \frac{1-p}{N_1+1-p}, & \text{if } p_1 = p_2 = p \\ \frac{(1-p_1)(1-\alpha)}{1-\frac{p_1}{p_2}\alpha^{N_1}} & \text{if } p_1 \neq p_2 \end{cases}. \quad (\text{A.51})$$

It can be shown that

$$\begin{aligned} Q_o(p_1, p_2, N_1) &< Q_1^{cl}(p_1, p_2, N_1, N_0, S), \\ &\quad S \leq \min(N_1, N_0), \\ Q_o(p_1, p_2, N_1) &< Q_2^{cl}(p_1, p_2, N_1, N_0, S), \\ &\quad N_0 < S \leq N_1, \\ Q_o(p_1, p_2, N_1) &= Q_2^{cl}(p_1, p_2, N_1, N_0, S), \\ &\quad N_1 < S \leq N_0, \\ Q_o(p_1, p_2, N_1) &< Q_3^{cl}(p_1, p_2, N_1, N_0, S), \\ &\quad S > \max(N_1, N_0). \end{aligned}$$

Therefore, the pair (N_0, S) is unimpeding, i.e.,

$$PR_{cl}(p_1, p_2, N_1, N_0, S) = PR_o(p_1, p_2, N_1)$$

if and only if $N_1 < S \leq N_0$. ■

Proof of Corollary 3: When the machines are asymptotically reliable in the sense that

$$p_i = 1 - \epsilon k_i, \quad i = 1, 2$$

where $0 < \epsilon \ll 1$ and $k_i > 0$ is independent of ϵ , it is easy to show that

$$\begin{aligned} Q_1^{cl}(p_1, p_2, N_1, N_0, S) &= Q^{eo}(p_1, p_2, S - 1) + O(\epsilon^2), \\ Q_2^{cl}(p_1, p_2, N_1, N_0, S) &= Q^{eo}(p_1, p_2, \min(N_1, N_0)) + O(\epsilon^2), \\ Q_3^{cl}(p_1, p_2, N_1, N_0, S) &= Q^{eo}(p_1, p_2, N_1 + N_0 - S + 1) + O(\epsilon^2) \quad (\text{A.52}) \end{aligned}$$

where

$$Q^{eo}(p_1, p_2, N_{eo}) = \begin{cases} \frac{1-p}{N_{eo}+1-p}, & \text{if } p_1 = p_2 = p, \\ \frac{(1-p_1)(1-\alpha)}{1-\frac{p_1}{p_2}\alpha^{N_{eo}}} & \text{if } p_1 \neq p_2. \end{cases}. \quad (\text{A.53})$$

Substituting (A.52) and (A.53) into (5)–(9), respectively, proves Corollary 3. ■

Proof of Theorem 2: When (24) occurs, m_1 is never starved and m_M is never blocked. Hence, the closed nature of the line does not impact the open line performance, which implies that (3.19) holds. ■

Proof of Theorem 3: Two cases are considered.

Case 1: $N_1 \neq N_0$. Under this condition, it follows from Theorem 1 that a closed line with two machines is S^+ -improvable if $S < \min(N_1, N_0) + 1$, i.e., $S_{\min}^* = \min(N_1, N_0) + 1$, and relationships (A.54) at the bottom of the page take place.

Clearly, $ST_1^{cl} > BL_1^{cl}$ and $ST_2^{cl} > BL_2^{cl}$. Therefore, for S^+ -improvable situation

$$ST_1^{cl} + ST_2^{cl} > BL_1^{cl} + BL_2^{cl}. \quad (\text{A.55})$$

Similarly, a closed line with two machines is S^- -improvable if $S > \max(N_1, N_0)$, i.e., $S_{\max}^* = \max(N_1, N_0)$, and

$$\begin{aligned} ST_1^{cl} &= 0 \\ BL_1^{cl} &= p_1(1 - p_2)Q^{cl}(p_2, p_1, N_0, N_1, S) \\ ST_2^{cl} &= 0 \\ BL_2^{cl} &= (1 - p_1)p_2Q^{cl}(p_1, p_2, N_1, N_0, S). \end{aligned} \quad (\text{A.56})$$

Therefore, for S^- -improvable situation

$$ST_1^{cl} + ST_2^{cl} < BL_1^{cl} + BL_2^{cl}. \quad (\text{A.57})$$

Case 2: $N_1 = N_0 = N$. Under this condition, it follows that

$$\begin{aligned} Q_1^{cl}(p_1, p_2, N_1 = N, N_0 = N, S = N) &< Q_3^{cl}(p_1, p_2, N_1 = N, N_0 = N, S = N + 1) \end{aligned}$$

and therefore

$$\begin{aligned} PR_{cl}(p_1, p_2, N_1 = N, N_0 = N, S = N) &> PR_{cl}(p_1, p_2, N_1 = N, N_0 = N, S = N + 1). \end{aligned}$$

In other words, the line is S^+ -improvable if $S < \min(N_1, N_0) = N$ and S^- -improvable if $S > \min$

$$\begin{aligned} ST_1^{cl} &= p_1Q^{cl}(p_2, p_1, N_0, N_1, S), \\ BL_1^{cl} &= \begin{cases} p_1(1 - p_2)Q^{cl}(p_2, p_1, N_0, N_1, S), & \text{if } S = N_1 < N_0, \\ 0, & \text{if } S < N_1, \end{cases} \\ ST_2^{cl} &= p_2Q^{cl}(p_1, p_2, N_1, N_0, S), \\ BL_2^{cl} &= \begin{cases} (1 - p_1)p_2Q^{cl}(p_1, p_2, N_1, N_0, S), & \text{if } S = N_0 < N_1, \\ 0, & \text{if } S < N_0. \end{cases} \end{aligned} \quad (\text{A.54})$$

$(N_1, N_0) = N$, i.e., $S_{\min}^* = S_{\max}^* = N$. Then, using again (A.54) and (A.56), we obtain that

$$\begin{aligned} ST_1^{cl} + ST_2^{cl} &> BL_1^{cl} + BL_2^{cl}, \text{ if } S < N, \\ ST_1^{cl} + ST_2^{cl} &< BL_1^{cl} + BL_2^{cl}, \text{ if } S > N \end{aligned} \quad (\text{A.58})$$

which completes the proof. ■

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REFERENCES

- [1] D. Jacobs and S. M. Meerkov, "A system-theoretic property of serial production lines—improvability," *Int. J. Syst. Sci.*, vol. 26, no. 5, pp. 755–785, 1995.
- [2] R. O. Onvural, "Survey of closed queueing networks with blocking," *ACM Comput. Surv.*, vol. 22, no. 2, pp. 83–121, Jun 1, 1990.
- [3] R. Suri and G. W. Diehl, "A variable buffer-size model and its use in analyzing closed queueing networks with blocking," *Manage. Sci.*, vol. 32, pp. 206–224, 1986.
- [4] X.-G. Liu, L. Zhuang, and J. A. Buzacot, R. Onvural and I. Akyildiz, Eds., "A decomposition method for throughput analysis in cyclic queues with production blocking," in *Queueing Networks with Finite Capacity*. Amsterdam, The Netherlands: Elsevier, 1993.
- [5] Y. Frein, C. Commault, and Y. Dallery, "Modeling and analysis of closed-loop production lines with unreliable machines and finite buffers," *IIE Trans.*, vol. 28, no. 7, pp. 545–554, 1996.
- [6] C.-H. Paik, H.-G. Kim, and H.-S. Cho, "Performance analysis for closed-loop production systems with unreliable machines and random processing times," *Comput. Ind. Eng.*, vol. 42, no. 2, pp. 207–220, 2002.
- [7] S. B. Gershwin and L. M. Werner, "An approximate analytical method for evaluating the performance of closed-loop flow systems with unreliable machines and finite buffers," *Int. J. Prod. Res.*, vol. 45, no. 14, pp. 3085–3311, 2007.
- [8] N. Maggio, A. Matta, S. B. Geshwin, and T. Tolio, "A decomposition approximation for three-machine closed-loop production systems," *IIE Trans.*, 2007.
- [9] Y. Dallery, Z. Liu, and D. Towsley, "Properties of fork/join queueing networks with blocking under various operating mechanisms," *IEEE Trans. Robot. Autom.*, vol. 13, no. 4, pp. 503–518, Aug. 1997.
- [10] J.-T. Lim and S. M. Meerkov, "On asymptotically reliable closed serial production lines," *Contr. Eng. Pract.*, vol. 1, no. 1, pp. 147–152, Feb. 1993.
- [11] D. S. Kim, "The equivalence of two-station closed and open serial production systems with finite buffers," *IIE Trans.*, vol. 30, no. 1, pp. 101–106, 1998.
- [12] D. S. Kim, D. M. Kulkarny, and F. Lin, "An upper bound for carriers in a three-workstation closed serial production system operating under production blocking," *IEEE Trans. Autom. Contr.*, vol. 47, pp. 1134–1138, 2002.
- [13] Z.-Y. Zhang, "Analysis and design of manufacturing systems with multiple-loop structures," Ph.D. dissertation, MIT, Cambridge, MA, 2006.
- [14] P. Glasserman and D. D. Yao, "Structured buffer-allocation problems," *Discrete Event Dyn. Syst.*, vol. 6, no. 1, pp. 9–41, 1996.
- [15] D. R. Staley, "General design rules for the allocation of buffers in closed serial production lines," M.S. thesis, Oregon State Univ., Corvallis, 2006.
- [16] C.-T. Kuo, J.-T. Lim, and S. M. Meerkov, "Bottlenecks in serial production lines: A system-theoretic approach," *Math. Prob. Eng.*, vol. 2, no. 3, pp. 233–276, 1996.



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