Bottlenecks in Markovian Production Lines: A Systems Approach

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Abstract—In this paper, a system-theoretic approach to bottlenecks in Markovian production lines is introduced and analyzed. The approach is based on the sensitivity of the system production rate to machines’ reliability parameters. Using this approach, definitions of bottlenecks are introduced, methods for their identification are developed, and their implications for production automation and preventative maintenance are discussed.

Index Terms—Markovian statistics of breakdowns, production bottlenecks, production systems, unreliable machines.

I. INTRODUCTION

Production lines are sets of machines and material handling devices arranged in the consecutive order so as to produce a desired product. From a system-theoretic perspective, production lines are discrete-event systems. Investigation of fundamental laws that govern their behavior has implication for both production automation and preventative maintenance.

One of the problems that impedes the performance of production lines is the lack of machines’ reliability: An unscheduled down-time of a machine may negatively affect the performance of all other machines, both up-stream and down-stream, blocking the former and starving the latter. Two basic models of machines reliability have been discussed in the literature: Bernoulli [1], [2] and Markovian [3]–[5]. Bernoulli model assumes that the status of a machine in each cycle (i.e., the time necessary to process one part) is determined by the process of Bernoulli trials. In Markovian model the state of a machine in a cycle is determined by a conditional probability, with the condition being the state of the machine in the previous cycle. This gives rise to a Markov process, which is why the term “Markovian production lines” is used.

Both models, Bernoulli and Markovian, reflect practical situations, however different ones. Bernoulli reliability model is more appropriate when down-time is small and comparable with the cycle time. This is often the case in assembly operations, where the down-time is due to quality problems (see [1]). Markovian model reflects operations where the down-time is due to mechanical failures which could be much longer than the cycle time, [3]–[5]. In this paper we address the Markovian model.

Intuitively, bottleneck (BN) of a production line is understood as a machine that impedes the system performance in the strongest manner. Identification of BNs and their improvement, either by automation or work restructuring, is considered as one of the most important problems in current manufacturing environment. In fact, the so-call Toyota Production System considers the continuous improvement as a part of three most important manufacturing programs: Just-in-Time (low inventories), Jidoka (quality), and Kaizen (continuous improvement) [6].

Unfortunately, not much is known, at least from the theoretical perspective, about BNs in production systems. Even the definition of a BN is unclear. For instance, some authors define the BN as the machine with the smallest isolation production rate (i.e. the production rate of the machine when no starvation and blockages are present). Others call the BN the machine with the largest inventory accumulated in front of it (see [7] and references therein). Both may not identify the machine that affects the bottom line, i.e. the system production rate, in the strongest possible manner (see [8]). This happens because the above definitions are local in nature and do not take into account the total system properties, such as the order of the machines in the production line, capacity of the material handling devices (buffers), etc. Therefore, a system-theoretic definition of a BN is necessary.

In the framework of Bernoulli lines, such a definition was introduced in [8], where a machine was referred to as the BN if the partial derivative of the system production rate with respect to machine’s isolation production rate was the largest. Since the system production rate depends on all—the order of the machines in the line, their performance and reliability parameters, the buffer structure, etc.—this definition captures the system nature of the BN.

This definition, however, is not applicable to Markovian production lines. The reason is that, unlike the Bernoulli case, where the machine’s production rate is defined by a single parameter, in the Markovian model the production rate is defined by two independent variables which characterize the up- and down-time. This makes it impossible to directly extend the Bernoulli case to the Markovian model. Therefore, the goal of this paper is to introduce definitions of BNs applicable to Markovian lines, suggest methods for their identification for the case of one- and two-machine production systems, and discuss implications of the results obtained for production automation and preventative maintenance.

The outline of the paper is as follows: In Section II, Markovian model of a production system is introduced, definitions of BNs are formulated, and the problems addressed are stated. Section III is devoted to the case of a single machine production system. Sections IV and V treat the two machines case. Finally, the conclusions are formulated in Section VI. The proofs are given in Appendices A and B.

II. DEFINITION AND PROBLEM FORMULATION

A. System Model

The following model of a production line is considered throughout this work.

1) The system consists of \( M \) machines arranged serially and \( M - 1 \) buffers separating each consecutive pair of machines.

2) Each machine \( m_i \) has two states: up and down. When up, the machine is capable of producing with the rate \( 1 \) part per unit of time (cycle); when the machine is down, no production takes place.

3) The up-time and the down-time of each machine \( m_i \) are random variables distributed exponentially with parameters \( \lambda_i \) and \( \mu_i \), respectively.
B. Bottleneck Definitions

Definition 2.1: Machine $m_i$ is the up-time bottleneck (UT-BN) if
\[
\frac{\partial \text{PR}}{\partial T_{\text{up}_i}} > \frac{\partial \text{PR}}{\partial T_{\text{up}_j}}, \quad j \neq i.
\]

It is the down-time bottleneck (DT-BN) if
\[
\left| \frac{\partial \text{PR}}{\partial T_{\text{down}_i}} \right| > \left| \frac{\partial \text{PR}}{\partial T_{\text{down}_j}} \right|, \quad j \neq i.
\]

Definition 2.2: Machine $m_i$ is the bottleneck (BN) if it is both UT-BN and DT-BN.

Definition 2.3: Let $m_i$ be the bottleneck machine. Then it is referred to as the up-time preventative maintenance bottleneck (UTPM-BN) if
\[
\frac{\partial \text{PR}}{\partial T_{\text{up}_i}} > \left| \frac{\partial \text{PR}}{\partial T_{\text{down}_i}} \right|.
\]

If the inequality is reversed, the bottleneck is referred to as the down-time preventative maintenance bottleneck (DTPM-BN).

Remark 2.2: Definitions 2.1–2.3 are formulated in terms of the up- and down-time, rather than $p_i$ and $r_i$ of model 2–6), because these are the variables used in practical situations on the factory floor. The differentiability of PR with respect to $T_{\text{up}_i}$ and $T_{\text{down}_i}$ is proved in Lemma A.1 of Appendix A. The absolute values of $\partial \text{PR}/\partial t$ are used because otherwise this number is negative: increase in $T_{\text{down}_i}$ leads to a decrease of PR.

The meaning of UT-BN and DT-BN is clear from Definition 2.1: A machine is UT-BN (or DT-BN) if an increase of its up-time (respectively, a decrease of it down-time) leads to the largest increase of the system production rate. The meaning of BN of Definition 2.2 is also obvious: A machine is the BN if both its up-time and its down-time are the most critical for the system performance.

III. ONE MACHINE CASE

If the system consists of a single machine, Definition 2.3 is the only applicable one, i.e., either UTPM-BN or DTPM-BN could exist.

If each of these takes place is quantified as follows:

Theorem 3.1: A single machine defined by assumptions 2)–3) is UTPM-BN if $T_{\text{up}} < T_{\text{down}}$: it is DTPM-BN if $T_{\text{down}} < T_{\text{up}}$. 

Proof: Follows immediately form (2.1) since

$$\frac{\partial PR}{\partial T_{down}} = \frac{T_{up}}{(T_{up} + T_{down})^2}$$

and

$$\frac{\partial PR}{\partial T_{up}} = \frac{T_{down}}{(T_{up} + T_{down})^2}.$$

Based on the above, we arrive at the following

**Bottleneck Indicator 3.1:** The smaller between the average up-time and average down-time of a machine defines its nature as the BN. If $T_{down} < T_{up}$, the primary attention of the preventive maintenance and automation should be given to the further decrease of the down-time; if $T_{up} < T_{down}$, the attention should be concentrated on the increase of the up-time.

Since in most practical situations $T_{down} < T_{up}$, the above Indicator basically states that reduction of the down-time is more efficient than a comparable increase of the up-time.

IV. TWO MACHINES CASE: EQUAL ISOLATION PRODUCTION RATES

Consider system 1)–6) with two machines, i.e., system $\{T_{up1}, T_{down1}, T_{up2}, T_{down2}, N_1\}$. Assume that

$$\frac{T_{up1}}{T_{down1}} = \frac{T_{up2}}{T_{down2}} \quad (4.1)$$

i.e., the efficiency of both machines are identical, however, the up- and down-time for each machine may be different. A question arises: Which one is the BN? The answer is as follows.

**Theorem 4.1:** Consider the production line $\{T_{up1}, T_{down1}, T_{up2}, T_{down2}, N_1\}$ and assume that (4.1) holds. Then $m_1$ is the BN if $T_{down1} < T_{down2}$; if the inequality is reversed, $m_2$ is the BN.

Proof: See Appendix A.

It is well known that, given a constant ratio between $T_{upj}$ and $T_{downi}$, the machine with the longer up- and down-time is more detrimental to the system’s production rate than that with a shorter up- and down-time (see, for instance, [5]). In view of this property, one might think that the BN is the machine with the longer up- and down-time. This, however, is not true, as the above Theorem states. The reason is an improvement of the machine with a shorter up- and down-time leads to a better utilization of the disturbance attenuation capabilities of the buffer than a comparable improvement of the machine with a longer up- and down-time. Therefore, an improvement of the “better” machine is the best for the system as a whole.

Assume now that (4.1) holds and, in addition

$$\frac{T_{upj}}{T_{downj}} = k > 1, \quad i, j = 1, 2, \quad i \neq j \quad (4.2)$$

i.e., $m_j$ is the BN. Is it UTPM-BN or DTPM-BN? The answer is in the following.

**Theorem 4.2:** Under assumption (4.2), $m_j$ is DTPM-BN if $T_{downj} < T_{upj}$; if $T_{downj}$ is sufficiently larger than $T_{upj}$, so that

$$\frac{T_{downj}}{T_{upj}} > \frac{k + 1}{k}$$

$m_j$ is UTPM-BN.

Proof: See Appendix A.

The above result is qualitatively in agreement with Theorem 3.1: The smaller between the up- and down-time of the BN defines its nature as far as the preventive maintenance is concerned.

From Theorems 4.1 and 4.2 follows

**Bottleneck Indicator 4.1:** In a production line with two machines of equal efficiency, the machine with the smaller down-time is the BN. If the down-time of this machine is smaller than its up-time, preventative maintenance and automation should be directed toward the decrease of the down-time. If the down-time is sufficiently larger than the up-time, preventative maintenance and automation should be directed toward the increase of the up-time.

Assume finally that (4.1) holds and, in addition

$$T_{up1} = T_{up2} = T_{up}, \quad T_{down1} = T_{down2} = T_{down} \quad (4.3)$$
i.e., both machines are identical as far as their reliability is concerned. In this situation, obviously, both machines are BNs but of which kind? The answer is given below:

**Theorem 4.3:** Consider $\{T_{up1}, T_{down1}, T_{up2}, T_{down2}, N_1\}$ and assume that (4.3) holds. Then both machines are DTPM-BN if

$$\frac{T_{up}}{T_{up} + T_{down}} > 0.5$$

both machines are UTPM-BN if

$$\frac{T_{up}}{T_{up} + T_{down}} < 0.4315.$$
The probability of manufacturing starvation, \( m_{s_i} \), is defined as

\[
\begin{align*}
\text{Definition 5.1:} & \quad m_{b_i} = \Pr \{ m_i \text{ is up at time } t \} \\
& \quad \cap \{ b_i \text{ is full at time } t \} \\
& \quad \cap \{ m_{i+1} \text{ fails to take parts from } b_i \text{ at time } t \}. \\
\end{align*}
\]

\[
\begin{align*}
\text{Definition 5.2:} & \quad m_{s_i} = \Pr \{ m_{i-1} \text{ fails to put parts into } b_{i-1} \text{ at time } t \} \\
& \quad \cap \{ b_{i-1} \text{ is empty at time } t \} \\
& \quad \cap \{ m_i \text{ is up at time } t \}. \\
\end{align*}
\]

In the case of Bernoulli lines, \( m_{s_i} \) and \( m_{b_i} \) were sufficient to characterize the BNs (see [8]). It turns out, however, that in Markovian case they should be modified appropriately to serve this purpose. Nevertheless, they play an important role in BN identification and, therefore, are characterized, in terms of the parameters of model 1)-6), below.

**Lemma 5.1.** For the serial production line 1)-6) with \( M = 2 \)

\[
\begin{align*}
m_{b_1} &= e_1 Q(p_2, r_2, p_1, r_1, N_1) \\
m_{s_2} &= e_2 Q(p_1, r_1, p_2, r_2, N_1)
\end{align*}
\]

(5.2)

where \( p_i = 1/T_{up_i} \) and \( r_i = 1/T_{down_i} \), \( i = 1, 2 \), \( Q \) is seen in (5.3), shown at the bottom of the page, and

\[
\begin{align*}
\varphi &= e_1(1-e_2) \\
\beta &= e_1(1-e_1) \\
\alpha &= (r_1 + r_2 + p_1 + p_2)/(p_1 r_2 - p_2 r_1) \\
\gamma &= (r_1 + r_2)/(p_1 + p_2)
\end{align*}
\]

**Proof:** See Appendix B.

**Lemma 5.2.** For the serial production line 1)-6) with \( M = 2 \) under assumption (4.1)

\[
m_{b_1} = m_{s_2}. \quad (5.4)
\]

**Proof.** See Appendix B.

The values of \( m_{b_1} \) and \( m_{s_2} \) calculated according Lemma 5.1, play a crucial role in DT-BN identification in production lines with unequal machines efficiency. Specifically,

**Bottleneck Indicator 5.1.** If \( m_{b_1} T_{up_1} T_{down_1} < m_{s_2} T_{up_2} T_{down_2} \), \( m_1 \) is the DT-BN. If \( m_{b_1} T_{up_1} T_{down_1} > m_{s_2} T_{up_2} T_{down_2} \), DT-BN is \( m_2 \).

**Remark 5.1.** Taking into account Lemma 5.2, it is easy to see that Bottleneck Indicator 5.1 is a generalization of Bottleneck Indicator 4.1 and the Bottleneck Indicator of [8]. Indeed, in the case of equal machines’ efficiency, machine \( m_1 \) is the BN if \( T_{up_1} T_{down_1} < T_{up_2} T_{down_2} \); if the inequality is reversed, \( m_2 \) is the BN. In the case of Bernoulli lines [8], \( m_1 \) is the BN if \( m_{b_1} < m_{s_2} \); if the inequality is reversed, \( m_2 \) is the BN. Thus, Bottleneck Indicator 5.1 is a generalization of the above two results.

**Numerical Justification:** At this time, Bottleneck Indicator 5.1 has been justified only numerically. Two typical examples are given in Fig. 2. In this figure, the first two rows of numbers below the machines are the average up-time and down-time, respectively. The other three rows of numbers show the values of \( m_{b_1} T_{up_1} T_{down_1}, m_{s_2} T_{up_2} T_{down_2} \), and \( |\Delta PR/\Delta T_{down_1}| \). The finite differences \( \Delta PR/\Delta T_{down_1} \), \( \forall i \), are numerical estimates of \( \partial PR/\partial T_{down_1} \) with the step \( \Delta T_{down_1} = 0.05 \cdot T_{down_1} \). Then, from Bottleneck Indicator 5.1, the bottlenecks in Fig. 2(a) and (b) are machines \( m_2 \) and \( m_1 \), respectively. In most systems investigated the bottlenecks identified using Bottleneck Indicator 5.1 and \( |\Delta PR/\Delta T_{down_1}| \) were the same. However, several counterexamples were also found. To illustrate the region where Bottleneck Indicator 5.1 does not work (Fig. 3), we use the parameters of the system in Fig. 2(a). In Fig. 3(a), \( T_{down_2} \) is changed from 1.176–0.08 and the bottleneck machine is switched from \( m_2 \) to \( m_1 \). As it follows from this figure, the range where Bottleneck Indicator 5.1 does not work is quite small. In Fig. 3(b), the efficiency of \( m_2 \) is used as the horizontal axis. We also obtain a similar result (i.e., the efficiency region where the Bottleneck Indicator 5.1 does not work is very small [about 0.01]). Based on the above, we conclude that it can be used as a tool for the DT-BN identification.

**VI. CONCLUSION**

Bottlenecks in Markovian production lines can be defined as partial derivatives of the system production rate with respect to machines’ up- and down-time. Three types of BNs are introduced:

1. up- and down-time BNs;
2. BNs;
3. UTPM- and DTPM-BNs.

Identification of each has implications for both the nature of preventative maintenance and production automation.

In “balanced” two-machine lines (i.e., in lines where machines have identical efficiency) the machine with the smaller down-time is the BN. In “unbalanced” two-machine lines, the DT-BN is the machine.
with the smallest value of $\alpha T_{up} T_{down}$, where $\alpha$ is the probability of blockage for the first machine and the probability of starvation for the second. Current work is centered on extensions of the results obtained to systems with arbitrary number of machines and on applications in the automotive industry.

APPENDIX A
PROOFS FOR SECTION IV

Due to space limitations, we present here and in Appendix B only the main steps of the proofs. The details can be found in [10].

The proof of Theorem 4.1 requires the expression for the production rate of two-machine system and the property of its differentiability with respect to the machines reliability parameters. The former follows from [9] in the form (A.1), as shown at the bottom of the page, where

$$\beta = \frac{(r_1 + r_2 + p_1 + p_2)(r_1 r_2 - p_2 r_1)}{(r_1 + r_2)(p_1 + p_2)}.$$  (A.2)

The latter is established by the following.

**Lemma A.1:** For all $0 \leq N_i < \infty$, production rate given by (A.1) is differentiable with respect to the remaining arguments $p_i, r_i, T_{up}, T_{down}$, and $T_{down}; i = 1, 2$.

**Proof—Step 1:** For the case of $p_1/r_1 = p_2/r_2$, using straightforward, but tedious, calculations, we show that the right derivative of PR with respect to $p_1$ is expressed as (A.3) as shown at the bottom of the page, where

$$\text{PR} = \left\{ \begin{array}{ll}
\frac{r_1 r_2}{(p_1 + r_1)(p_2 + r_2)} \left[ \frac{p_1 (p_2 + r_2) - p_2 (p_1 + r_1) e^{-N_1}}{p_1 r_2 - p_2 r_1 e^{-N_1}} \right], & p_1 \neq p_2 \\
\frac{r_2^2 (r_1 + r_2) + N_1 r_1 r_2 (p_2 + r_2)^2}{(p_2 + r_2)^2 (r_1 + r_2 + N_1 r_1 (p_2 + r_2))}, & p_1 = p_2 \\
\end{array} \right.$$

(A.1)

$$\lim_{\Delta p \to 0^+} \frac{\text{PR}(p_1 + \Delta p, r_1, p_2, r_2, N_1) - \text{PR}(p_1, r_1, p_2, r_2, N_1)}{\Delta p} = r_1 r_2 \left[ \frac{p_1 g_1 - p_2 (p_1 + r_1) g_1 - f_1 (d_1 - p_2 r_1 a_1) [p_1 + r_1] - f_1 g_1}{(p_1 + r_1)^2 (p_2 + r_2) g_1^2} \right]$$  (A.3)
Step 2: Analogously, the left derivative PR is (A.5), as shown at the bottom of the page, where \( f_1, g_1, n_1, d_1, \) and \( a_1 \) are given by (A.4).

Step 3: From (A.3) and (A.5), it follows that the production rate PR is differentiable with respect to \( p_1 \). Analogously, we show that PR is differentiable with respect to with respect to \( r_1, p_2, \) and \( r_2 \).

Step 4: Since

\[
\frac{\partial PR}{\partial T_{ap_1}} = \frac{\partial PR}{\partial p_1} \frac{\partial p_1}{\partial T_{ap_1}} = \frac{\partial PR}{\partial p_1} (-p_1^2), \quad i = 1, 2
\]

\[
\frac{\partial PR}{\partial T_{down_1}} = \frac{\partial PR}{\partial r_i} \frac{\partial r_i}{\partial T_{down_1}} = \frac{\partial PR}{\partial r_i} (-r_i^2), \quad i = 1, 2
\]

it follows that the production rate PR is differentiable with respect to \( T_{ap_1} \) and \( T_{down_1} \), \( i = 1, 2 \).

Step 5: For \( p_1/r_1 \neq p_2/r_2 \), Steps 1–4 are repeated to establish the differentiability of PR in this case as well.

**Proof of Theorem 4.1— Step 1:** From (A.1), we show (A.7), as seen on the bottom of the page, where

\[
f_1 = 1 + p_2(p_1 + r_1) r_1 + r_2 + p_1 + p_2 N_1 \]
\[
g_1 = 1 + p_2 r_1 + r_2 + p_1 + p_2 N_1 \]
\[
n_1 = N_1 p_2 \]
\[
d_1 = \frac{(p_1 + p_2)(r_1 + r_2 + p_1 + p_2) - (p_1 + r_1)(r_1 + r_2)}{(r_1 + r_2)(p_1 + p_2)^2} \]
\[
a_1 = \frac{(r_1 + r_2 + p_1 + p_2)^2 N_1^2}{2(r_1 + r_2)^2(p_1 + p_2)^2} \]
\[
i, j = 1, 2, \quad i \neq j
\]

**Step 2:** Under assumption (4.1), from the above expressions and (A.6), we show that

\[
\frac{\partial PR}{\partial T_{down_1}^i} = \frac{\partial PR}{\partial r_i} \cdot \frac{\partial r_i}{\partial T_{down_1}^i} = \frac{\partial PR}{\partial r_i} \cdot (-r_i^2), \quad i = 1, 2
\]

\[
\frac{\partial PR}{\partial T_{up_1}} = \frac{\partial PR}{\partial p_1} \cdot \frac{\partial p_1}{\partial T_{up_1}} = \frac{\partial PR}{\partial p_1} \cdot (-p_1^2) \]

\[
= \frac{2(p_1 + r_1)^2(p_2 + r_2)(r_1 + r_2)^2(p_1 + p_2)^2 g_i^2}{N_1^2 p_2 r_1 (p_2 + r_2)(r_1 + r_2 + p_1 + p_2)^2}
\]

\[
+ 2(r_1 + r_2)^2(p_1 + p_2)^2 + 2N_1 p_2 (r_1 + r_2)
\]

\[
\cdot \left[ (p_1(r_1 + r_2)p_2 + r_2)(r_1 + r_2 + p_1 + p_2)^2 + (r_1 + r_2 + p_1 + p_2)^2 \right]
\]

where \( k = T_{up_1}/T_{ap_1} = T_{down_2}/T_{down_1} = p_1/p_2 = r_1/r_2. \)

If \( k > 1 \), i.e., the up- and down-time of \( m_1 \) are shorter than those of \( m_2 \), then, since \( g_1 > 0 \).

\[
\frac{\partial PR}{\partial T_{up_1}} > \frac{\partial PR}{\partial T_{down_2}} \]

Thus, the machine with the shorter up- and down-time is the UP-BN.

**Step 3:** Steps 1 and 2 are repeated for \( T_{down_i} \), \( i = 1, 2 \), to obtain

\[
\frac{\partial PR}{\partial T_{down_1}^i} > \frac{\partial PR}{\partial T_{down_2}^i} \]

\[
= \frac{2(p_1 + r_1)^2(p_2 + r_2)(r_1 + r_2)^2(p_1 + p_2)^2 g_i^2}{N_1^2 p_2 r_1 (p_2 + r_2)(r_1 + r_2 + p_1 + p_2)^2}
\]

\[
+ 2(r_1 + r_2)^2(p_1 + p_2)^2 + 2N_1 p_2 (r_1 + r_2)
\]

\[
\cdot \left[ (p_1(r_1 + r_2)p_2 + r_2)(r_1 + r_2 + p_1 + p_2)^2 + (r_1 + r_2 + p_1 + p_2)^2 \right]
\]

If \( k > 1 \), we have

\[
\left| \frac{\partial PR}{\partial T_{down_1}^i} \right| > \left| \frac{\partial PR}{\partial T_{down_2}^i} \right| \]

i.e., the machine with the shorter up- and down-time is the DT-BN.

**Step 4:** From (A.10) and (A.12), it follows that the BN is the machine with the shorter up- and down-time.

**Proof of Theorem 4.2— Step 1:** From Theorem 4.1 and Assumption (4.2), we have \( \frac{\partial PR}{\partial T_{up_j}} \geq \frac{\partial PR}{\partial T_{ap_1}} \), and | \( \frac{\partial PR}{\partial T_{down_j}} \mid < \mid \frac{\partial PR}{\partial T_{down_i}} \mid, i, j = 1, 2, i \neq j \), i.e., machine \( j \) is the BN.
from (A.1), we show that

\[ k \alpha_1 \alpha_2 = \frac{\kappa e^{-\beta h_1}}{(p_1 + p_2)(p_1 + p_2 + 1)} \]

or UTPM-BN, we show that

\[ \alpha_1 \alpha_2 \]

It follows that if \( \alpha_1 \alpha_2 \)

\[ \beta \]

From the above expressions, if \( r_i > p_j \) or \( T_{up} > T_{down} \), then we have

\[ \frac{\partial PR}{\partial T_{down}} - \frac{\partial PR}{\partial T_{up}} = 0 \]  

where \( g_j \) is in (A.8) and \( k = \frac{p_j}{p_i} = \frac{r_j}{r_i} \).

Step 3: From the above expressions, if \( r_i > p_j \) (i.e., \( r_j > p_j \) or \( T_{up} > T_{down} \)), then we have

\[ \frac{\partial PR}{\partial T_{down}} - \frac{\partial PR}{\partial T_{up}} = 0 \]  

i.e., \( m_j \) is DTPM-BN. If \( r_i < p_j \) and \( k(r_i - p_j) + r_i < 0 \) which implies that \( T_{down} > T_{up} \) (i.e., \( T_{down,j} > T_{up,j} \)) and \( T_{down,j} > T_{up,j} \) \( (k + 1)/k \) (i.e., \( T_{down,j} > T_{up,j} \) \( (k + 1)/k \) \( T_{up} \)), we have

\[ \frac{\partial PR}{\partial T_{down}} < \frac{\partial PR}{\partial T_{up}} \]  

(A.15)

It follows that if \( T_{down,j} > T_{up,j} \) \( (k + 1)/k \), \( m_j \) is UTPM-BN.

Proof of Theorem 4.3—Step 1: Under the Assumption (4.3), from (A.1), we show that

\[ PR = \frac{2T_{up}[2T_{up}T_{down} + N_1(T_{up} + T_{down})^2]}{2(T_{up} + T_{down})^2[2T_{up}T_{down} + N_1(T_{up} + T_{down})]} \]  

(A.16)

Step 2: From the above expression, it follows:

\[ \frac{\partial PR}{\partial T_{up}} - \frac{\partial PR}{\partial T_{down}} = \frac{T_{up} + T_{down}}{(T_{up} + T_{down})^2[2T_{up}T_{down} + N_1(T_{up} + T_{down})]^2} \]

\[ \cdot \left( [N_1^2(T_{up} + T_{down})^2(T_{down} - T_{up}) + 4 N_1 T_{up}^2 T_{down} \cdot T_{down}(T_{up} + T_{down})(T_{down} - T_{up}) + 8T_{up} \cdot T_{down}^2(T_{down} - T_{up}) \right) \]  

(A.17)

Step 3: The threshold \( \tau \) (i.e., the efficiency such that \( \frac{\partial PR}{\partial T_{down}} = \frac{\partial PR}{\partial T_{up}} \)) is defined by

\[ \frac{\partial PR}{\partial T_{down}} - \frac{\partial PR}{\partial T_{up}} = 0. \]  

(A.18)

Step 4: From the above expression, for the minimum \( \tau \), we show that

\[ N_1 = \frac{-2T_{up} T_{down}(T_{down} - 2 T_{up})}{(T_{up} + T_{down})^2(T_{down} - T_{up})} \]  

(A.19)

Step 5: From (A.18) and (A.19), it follows that the threshold values of \( T_{up} \) and \( T_{down} \) are defined by the following equation:

\[ T_{down}^3 - 1.5T_{up} T_{down} + T_{up}^3 T_{down} - T_{up}^3 = 0. \]  

(A.20)

Solving this equation by Matlab, we obtain

\[ T_{down} = 1.317182646506777T_{up}. \]  

(A.21)

Moreover, substituting (A.21) into (A.19), we have

\[ N_1 = 0.8T_{down}. \]

Step 6: Therefore, from (A.21), if \( T_{up}/(T_{up} + T_{down}) < 0.4315 \), both machines are UT-BNs. From (A.17), if \( T_{up} > T_{down} \) (i.e., \( T_{up}/(T_{up} + T_{down}) > 0.5 \)), both machines are DT-BNs.

APPENDIX B

PROOFS FOR SECTION V

Consider the system defined by assumption 1–6 with \( M = 2 \). Introduce the stationary probability distribution, \( Y_{1,\alpha_1\alpha_2}(h_1) \), as follows:

\[ Y_{1,\alpha_1\alpha_2}(h_1) = \text{Prob} \{ \text{buffer } b_1 \text{ contains } h_1 \text{ parts} \} \]

\[ \text{and machines } m_1 \text{ and } m_2 \text{ are in the states of } \alpha_1 \text{ and } \alpha_2 \}

where \( 0 \leq h_1 \leq N_1 \), and \( \alpha_1 = u \) (i.e., \( m_1 \) is up) or \( d \) (i.e., \( m_1 \) is down), \( i = 1, 2 \). The PR of this two machines-one buffer system can be calculated using the following:
Lemma B.1: For the serial production line 1)–6) with $M = 2$

$$\begin{align*}
PR &= e_2 - Y_{1,u}(0) \\
&= e_1 - Y_{1,u}(N_1) \\
&= e_1[1 - Q(p_1, r_2, p_2, N_1)] \\
&= e_1[1 - Q(p_2, r_2, p_1, r_1, N_1)] \\
\end{align*}$$

(B.1)

where function $Q(x_1, y_1, x_2, y_2, N_1)$ is defined in (5.3).

Proof—Step 1: Let $X_{1,a_1=2}(h_1)$ where $0 \leq h_1 \leq N_1$, and $a_i = u$ (i.e., $m_i$ is up) or $d$ (i.e., $m_i$ is down), $i = 1, 2$, be the stationary density of buffer occupancy. The densities $X_{1,a_1=2}(h_1)$ and probabilities $Y_{1,a_1=2}(0)$ and $Y_{1,a_1=2}(N_1)$ are calculated in [9] and summarized in Table I.

Step 2: Using Table I, we show that

$$PR = e_2 - Y_{1,d}(0) \\
= e_1 - Y_{1,u}(N_1).$$

Step 3: Using (5.3), (A.1), and the above expressions, we show that

$$PR = e_2[1 - Q(p_1, r_1, p_2, r_2, N_1)] \\
= e_1[1 - Q(p_2, r_2, p_1, r_1, N_1)].$$

Proof of Lemma 5.1: From Definition 5.1

$$mb_1 = \text{Prob}\{\{m_1 \text{ is up at time } t\} \cap \{b_1 \text{ is empty at time } t\} \cap \{m_2 \text{ is up at time } t\}\} \\
= Y_{1,u}(N_1).$$

From Lemma B.1, we have

$$mb_1 = e_1 Q(p_2, r_2, p_1, r_1, N_1).$$

According to Definition 5.2

$$m_{s_2} = \text{Prob}\{\{m_1 \text{ is down at time } t\} \cap \{b_1 \text{ is empty at time } t\} \cap \{m_2 \text{ is up at time } t\}\} \\
= Y_{1,d}(0).$$

Using Lemma B.1, we obtain

$$m_{s_2} = e_2 Q(p_1, r_1, p_2, r_2, N_1).$$

Proof of Lemma 5.2: Follows immediately from Lemma B.1, since $e_1 = e_2$ implies that $m_{s_2} = mb_1$.

REFERENCES


