DT-Bottlenecks in Serial Production Lines: Theory and Application

Shu-Yin Chiang, *Member, IEEE*, Chih-Tsung Kuo, *Member, IEEE*, and Semyon M. Meerkov, *Fellow, IEEE*

***Abstract*—The bottleneck (BN) of a production line is a machine that impedes the system's performance in the strongest manner. It has been shown in the previous work that the so-called downtime bottlenecks (DT-BNs) are of main industrial importance. In this paper, a method for DT-BN identification is developed in the framework of serial production lines with unreliable (Markovian) machines and finite buffers. The identification tool derived is based on the probabilities of machine blockages and starvations. To evaluate these probabilities, a novel aggregation technique is proposed. The results obtained are applied to the design of a production line at an automotive component plant.**

***Index Terms*—Bottleneck, downtime, identification, production lines.**

I. Introduction

PRODUCTION lines are sets of machines arranged so as to produce a finished product or a component of a product. Machines are typically unreliable and experience random breakdowns, which lead to unscheduled downtime and loss of production. Breakdowns of a machine affects all other machines in the system, causing blockage of those upstream and starvation of those downstream. To minimize these perturbations, machines are separated by finite buffers, the empty space of which protects against blockage and the full against starvation. Thus, production lines may be modeled as sets of machines and buffers connected according to a certain topology. In this paper, we study systems with linear topology, i.e., serial production lines (see Fig. 1, where the circles are the machines and the rectangles are the buffers).

Statistics of the machine downtime depend on the type of manufacturing operations involved. Specifically, in assembly operations, where the downtime is often due to quality problems, most interruptions are quite short and comparable with the cycle time (i.e., the time necessary to accomplish an operation). In machining operations, the downtime is typically due to mechanical breakdowns and is much longer than the cycle time. Accordingly, breakdowns in assembly lines are often modeled by the process of Bernoulli trials, i.e., during each cycle time, a machine, which is not blocked and not starved, produces a part with a fixed probability and fails to do so with the complementary probability. In machining operations, machine reliability is often modeled by a Markov process, i.e., if a machine is up (respectively, down), it goes down (respectively, up) during an infinitesimal time interval with a fixed probability rate. This gives rise to uptime and downtime being distributed exponentially and to Markovian, rather than Bernoulli, description of machine reliability. Serial production lines with Bernoulli machines have been analyzed in [1] and [2] and with Markovian machines in the majority of publications on production lines (see [3]–[9] for the origin and recent status of this literature). The current paper addresses the Markovian case.

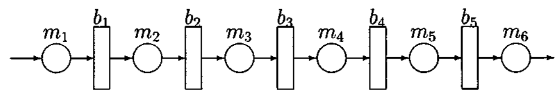


Fig. 1. Serial production line.

Due to machine breakdowns, production systems often produce at a level far below their capacity, i.e., the amount of production that can be obtained when no unscheduled downtime takes place. For instance, it is not uncommon in the automotive industry that production lines in machining operations produce at the level of 60%–70% of their capacity. In this situation, identification and improvement of a machine, which is most responsible for these losses, is an important problem of production line management and control. Such a machine is referred to as the bottleneck (BN).

The notion of BN in Bernoulli lines has been formalized and analyzed in [10]. It has been shown that if a machine is blocked more often than it is starved, the BN is downstream of this machine; otherwise, the BN is upstream. The problem of BN identification in Markovian lines has been formulated in [11]. Several notions of BNs have been introduced, depending on whether the uptime or the downtime is addressed. These were referred to as the uptime BN (UT-BN) and downtime BN (DT-BN), respectively. It has been shown that if the production rate of each machine in isolation is above 50% of its capacity (which is almost always the case on the factory floor), decreasing downtime results in a larger increase of the system production rate than a corresponding increase of the uptime. This leads to the conclusion that DT-BNs are of main industrial importance (see [11] for details).

The problem of DT-BN identification in two-machine lines was analyzed in [11]. No results for longer lines have been obtained. *The goal of this paper is to provide a tool for DT-BN identification in longer lines and illustrate its application by a recent case study*.

Manuscript received February 26, 1999; revised July 7, 2000. This paper was recommended for publication by Associate Editor J. Tsai and Editor N. Viswanadham upon evaluation of the reviewers' comments. This paper was presented in part at the IEEE CDC, Tampa, FL, December 1998.

S.-Y. Chiang is with the Department of Computer Communication Engineering, Ming-Chuan University, Taoyuan County, Taiwan, R.O.C.

C.-T. Kuo, deceased, was with the Department of Electrical Engineering, Tatung Institute of Technology, Taipei, Taiwan, R.O.C.

S. M. Meerkov is with the Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI 48109-2122 USA (e-mail: smm@eecs.umich.edu).

Publisher Item Identifier S 1042-296X(00)09778-0.

Results obtained in [11] have been derived using closed-form expressions for the system performance measures as functions of the machine and buffer parameters. Unfortunately, such expressions are not available for longer lines. Although efficient recursive approximations do exist [4]–[9], none is proven to converge. Moreover, existing approximations are not directly applicable to production systems considered in this work (see Section II). Therefore, in this paper, we first develop a novel aggregation procedure for performance analysis of Markovian production lines, prove its convergence, evaluate its accuracy, and then apply it to DT-BN identification.

The outline of the paper is as follows. In Section II, the model of the production line under consideration is introduced and the problem of DT-BN identification is formulated. In Section III, performance evaluation issues are discussed. Section IV presents the DT-BN identification technique. The case study is described in Section V. The conclusions are given in Section VI. The proof of convergence is included in the Appendix.

II. Problem Formulation

The following model of a serial production line is considered throughout this work.

1) The system consists of M machines arranged serially and $M-1$ buffers separating each pair of machines.
2) Each machine m_i has two states: up and down. When up, the machine is capable of producing with the rate 1 part per unit of time (cycle); when the machine is down, no production takes place.
3) The uptime and the downtime of each machine m_i are random variables distributed exponentially with parameters p_i and r_i, respectively.
4) Each buffer b_i is characterized by its capacity $N_i < \infty$, $1 \le i \le M-1$.
5) Machine m_i is starved at time t if buffer b_{i-1} is empty and m_{i-1} fails to put a part into b_{i-1} at time t; machine m_1 is never starved.
6) Machine m_i is blocked at time t if buffer b_i is full and m_{i+1} fails to take a part from b_i at time t; machine m_M is never blocked.

A production line defined by 1)–6) is denoted as $\{p_1, r_1, \ldots, p_M, r_M, N_1, \ldots, N_{M-1}\}$.

Remark 2.1: Due to assumption 3), the average uptime and downtime of the machines are

$$T_{\mathrm{up}_i} = \frac{1}{p_i} \qquad T_{\mathrm{down}_i} = \frac{1}{r_i}, \qquad i = 1, \ldots, M.$$

Therefore, the production rate in isolation of each machine (i.e., the average number of parts produced per unit of time if no starvation or blockage takes place) is

$$\mathrm{PR}_{\mathrm{iso}_i} = \frac{T_{\mathrm{up}_i}}{T_{\mathrm{up}_i} + T_{\mathrm{down}_i}} = \frac{1}{1+\dfrac{T_{\mathrm{down}_i}}{T_{\mathrm{up}_i}}}, \qquad i = 1, \ldots, M. \tag{2.1}$$

The isolation production rate is often referred to as the machine efficiency and is denoted as e_i, $i = 1, \ldots, M$. Note that the machine efficiency depends only on the ratio of T_{up_i} and T_{down_i}, rather than on their absolute values. □

Production rate of the line $\{p_1, r_1, \ldots, p_M, r_M, N_1, \ldots, N_{M-1}\}$ is the average number of parts produced by the last machine m_M per cycle. Given model 1)–6), the production rate $\tilde{\mathrm{PR}}$ is a function of the machine and buffer parameters

$$\tilde{\mathrm{PR}} = \tilde{\mathrm{PR}}(p_1, r_1, p_2, r_2, \ldots, p_M, r_M, N_1, \ldots, N_{M-1}). \tag{2.2}$$

We use this function to define the DT-BN.

Definition 2.1 [11]: Machine m_i is the DT-BN if

$$\left|\frac{\partial \tilde{\mathrm{PR}}}{\partial T_{\mathrm{down}_i}}\right| > \left|\frac{\partial \tilde{\mathrm{PR}}}{\partial T_{\mathrm{down}_j}}\right| \qquad \forall j \ne i. \tag{2.3}$$

□

Remark 2.2: Throughout this paper, symbols with the "$\sim$" denote the exact values of the appropriate quantity. The respective approximations, to be introduced below, are denoted by the same symbols, but without the "$\sim$." □

Unfortunately, direct identification of DT-BN using Definition 2.1 is impossible. The reason is twofold. First, the derivatives of $\tilde{\mathrm{PR}}$ cannot be measured on the factory floor during the normal system operation. Second, in most cases, they cannot be calculated analytically as well, since, as it has been pointed out above, even the calculation of $\tilde{\mathrm{PR}}$ itself for systems with more than two machines is impossible, let alone the calculation of its derivative. In addition, the perturbation analysis [12]–[14] cannot be applied either since no discrete event model of the system is assumed to be available, and moreover, parameters of the machines and buffers may be unknown. Therefore, the identification method has to be an indirect one. More specifically, we are seeking a DT-BN identification tool, which is based on the data available on the factory floor through real-time measurement (such as average uptime and downtime, starvation and blockage time, etc.) or on the data that can be constructively calculated using the machine and buffer parameters (but without using discrete event simulations). We refer to this tool as the **DT-BN Indicator**. The problem, then, addressed in this paper is as follows: *Given a production system defined by assumptions 1)–6), derive an indicator for DT-BN identification.*

III. Performance Evaluation in Serial Lines with Markovian Machines

A. Performance Measures

Performance measures of interest in this work are the system production rate and the probabilities of machine manufacturing starvation and blockage. The former is defined in Section II [see the text above expression (2.2)]. The latter are defined as

$$\begin{aligned}
\tilde{ms}_i = \mathrm{Prob}(&\{m_{i-1} \text{ fails to put a part into } b_{i-1} \text{ at time } t\}\\
&\cap \{b_{i-1} \text{ is empty at time } t\}\\
&\cap \{m_i \text{ is up at time } t\})\\
\tilde{mb}_i = \mathrm{Prob}(&\{m_i \text{ is up at time } t\}\\
&\cap \{b_i \text{ is full at time } t\}\\
&\cap \{m_{i+1} \text{ fails to take a part from } b_i \text{ at time } t\}).
\end{aligned}$$

These probabilities are functions of the machine and buffer parameters

$$\tilde{ms}_i = \tilde{ms}_i(p_1, r_1, p_2, r_2, \ldots, p_M, r_M, N_1, \ldots, N_{M-1})$$
$$\tilde{mb}_i = \tilde{mb}_i(p_1, r_1, p_2, r_2, \ldots, p_M, r_M, N_1, \ldots, N_{M-1}).$$

A recursive procedure for estimating these performance measures is given next.

B. Recursive Procedure and Estimates of Performance Measures

A recursive procedure considered in this work iterates the values of p_i, r_i, and N_i to produce a sequence of numbers $p_i^b(s)$, $r_i^b(s)$, $p_i^f(s)$, and $r_i^f(s)$, $s = 0, 1, 2, \ldots$, according to the rule (3.1), shown at the bottom of the page, with boundary conditions

$$p_1^f(s) = p_1, \quad r_1^f(s) = r_1, \quad p_M^b(s) = p_M$$
$$r_M^b(s) = r_M, \qquad s = 0, 1, 2, \ldots$$

and initial conditions

$$p_i^f(0) = p_i, \quad r_i^f(0) = r_i, \qquad i = 2, \ldots, M-1$$

where function Q is given in (3.2), shown at the bottom of the page.

There are two principal components of this procedure: backward and forward aggregation (denoted by superscripts b and f, respectively). In the backward aggregation, the last two machines m_M and m_{M-1} are aggregated into a single machine m_{M-1}^b, defined by parameters p_{M-1}^b and r_{M-1}^b. Then, machine m_{M-1}^b is aggregated with m_{M-2} to result in m_{M-2}^b, defined by p_{M-2}^b and r_{M-2}^b, and so on until all machines are aggregated into m_1^b. In the forward aggregation, the first machine m_1 is aggregated with m_2^b to produce m_2^f with parameters p_2^f and r_2^f. Then, m_2^f is aggregated with m_3^b to result in m_3^f, and so on until all machines are aggregated into m_M^f. Then, the process is repeated again.

The question of convergence of the resulting sequences $p_i^b(s)$, $r_i^b(s)$, $p_i^f(s)$, and $r_i^f(s)$, $s = 0, 1, \ldots$, is answered in the following theorem.

Theorem 3.1: Recursive procedure (3.1) is convergent and, therefore, the following limits exist:

$$\lim_{s\to\infty} p_i^f(s) =: p_i^f, \quad \lim_{s\to\infty} p_i^b(s) =: p_i^b, \quad \lim_{s\to\infty} r_i^f(s) =: r_i^f$$
$$\lim_{s\to\infty} r_i^b(s) =: r_i^b, \qquad i = 1, \ldots, M. \tag{3.3}$$

Moreover, the following relationship holds:

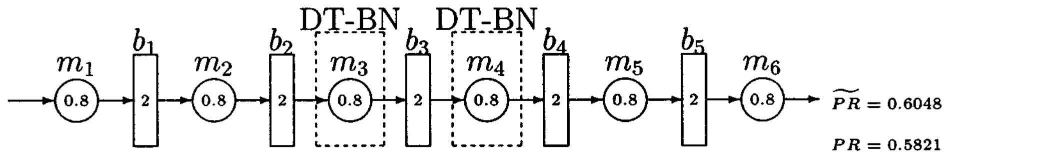
$$\frac{r_M^f}{p_M^f} = \frac{r_1^b}{p_1^b}. \tag{3.4}$$

Proof: See the Appendix.

The limits in (3.3) can be used to define estimates of performance measures for line 1)–6).

$$p_i^b(s+1) = \frac{p_i}{1 - Q(p_{i+1}^b(s+1), r_{i+1}^b(s+1), p_i^f(s), r_i^f(s), N_i)}, \qquad 1 \le i \le M-1$$
$$r_i^b(s+1) = \frac{1}{\dfrac{Q(p_{i+1}^b(s+1), r_{i+1}^b(s+1), p_i^f(s), r_i^f(s), N_i)}{p_i} + \dfrac{1}{r_i}}, \qquad 1 \le i \le M-1$$
$$p_i^f(s+1) = \frac{p_i}{1 - Q(p_{i-1}^f(s+1), r_{i-1}^f(s+1), p_i^b(s+1), r_i^b(s+1), N_{i-1})}, \qquad 2 \le i \le M$$
$$r_i^f(s+1) = \frac{1}{\dfrac{Q(p_{i-1}^f(s+1), r_{i-1}^f(s+1), p_i^b(s+1), r_i^b(s+1), N_{i-1})}{p_i} + \dfrac{1}{r_i}}, \qquad 2 \le i \le M \tag{3.1}$$

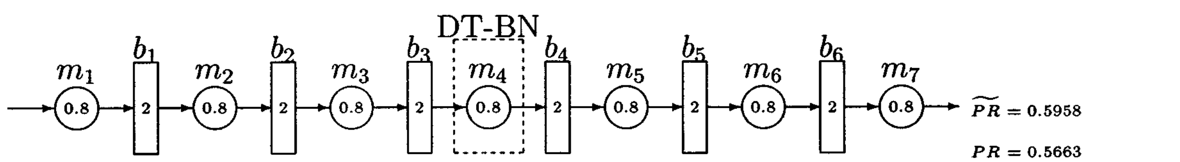
$$Q(p_1, r_1, p_2, r_2, N_1) = \begin{cases} \dfrac{(1-e_1)(1-\phi)}{1 - \phi e^{-\beta N_1}}, & \dfrac{p_1}{r_1} \neq \dfrac{p_2}{r_2} \\[2ex] \dfrac{p_1(p_1+p_2)(r_1+r_2)}{(p_1+r_1)[(p_1+p_2)(r_1+r_2) + p_2 r_1(p_1+p_2+r_1+r_2)N_1]}, & \dfrac{p_1}{r_1} = \dfrac{p_2}{r_2} \end{cases} \tag{3.2}$$
$$e_i = \frac{r_i}{p_i + r_i}, \qquad i = 1, 2$$
$$\phi = \frac{e_1(1-e_2)}{e_2(1-e_1)}$$
$$\beta = \frac{(r_1+r_2+p_1+p_2)(p_1 r_2 - p_2 r_1)}{(r_1+r_2)(p_1+p_2)}$$



$\frac{\widetilde{PR}-PR}{\widetilde{PR}} \cdot 100\% = 3.75\%$

T_{up_i} :	5	5	5	5	5	5
T_{down_i} :	1.25	1.25	1.25	1.25	1.25	1.25
$\widetilde{ms}_i$:	0	0.0647	0.822	0.1126	0.1454	0.195
ms_i :	0	0.0573	0.099	0.1357	0.173	0.2179
$\lvert\widetilde{ms}_i - ms_i\rvert/\widetilde{ms}_i \cdot 100\%$:	--	11.44	20.44	20.52	18.98	11.74
$\widetilde{mb}_i$:	0.1953	0.1462	0.1117	0.0817	0.0467	0
mb_i :	0.2179	0.1730	0.1357	0.099	0.057	0
$\lvert\widetilde{mb}_i - mb_i\rvert/\widetilde{mb}_i \cdot 100\%$:	11.57	18.33	21.49	21.18	20.06	--
$\tilde{\sigma}_j$:	0	0.3574	0.5973	0.7839	0.9652	1.2189
$\tilde{\beta}_i$:	1.2203	0.9701	0.7781	0.5942	0.3571	0
σ_i :	0	0.4569	0.7451	0.9676	1.1644	1.3617
β_i :	1.3617	1.1644	0.9676	0.7451	0.4569	0
$\lvert\Delta\widetilde{PR}/\Delta T_{down_i}\rvert$:	0.0409	0.0447	0.049	0.0533	0.0488	0.0392
$\lvert\Delta PR/\Delta T_{down_i}\rvert$:	0.0332	0.0497	0.058	0.0582	0.0497	0.0332

Fig. 2. System 1.



$\frac{\widetilde{PR}-PR}{\widetilde{PR}} \cdot 100\% = 4.95\%$

T_{up_i} :	5	5	5	5	5	5	5
T_{down_i} :	1.25	1.25	1.25	1.25	1.25	1.25	1.25
$\widetilde{ms}_i$:	0	0.0445	0.0744	0.101	0.125	0.1577	0.2034
ms_i :	0	0.0553	0.0942	0.1269	0.1581	0.1916	0.2337
$\lvert\widetilde{ms}_i - ms_i\rvert/\widetilde{ms}_i \cdot 100\%$:	--	24.27	26.61	25.64	26.48	21.5	14.9
$\widetilde{mb}_i$:	0.2042	0.1612	0.1302	0.1072	0.0779	0.0462	0
mb_i :	0.2337	0.1916	0.1581	0.1269	0.0942	0.0553	0
$\lvert\widetilde{mb}_i - mb_i\rvert/\widetilde{mb}_i \cdot 100\%$:	14.45	18.86	21.43	18.38	20.92	19.7	--
$\tilde{\sigma}_j$:	0	0.3481	0.5551	0.7291	0.8659	1.0457	1.0457
$\tilde{\beta}_i$:	1.276	1.067	0.8973	0.7668	0.5773	0.3596	0
σ_i :	0	0.4547	0.734	0.9427	1.1198	1.2865	1.4605
β_i :	1.4605	1.2865	1.1198	0.9427	0.734	0.4547	0
$\lvert\Delta\widetilde{PR}/\Delta T_{down_i}\rvert$:	0.0384	0.0418	0.0522	0.0592	0.0535	0.049	0.0386
$\lvert\Delta PR/\Delta T_{down_i}\rvert$:	0.0265	0.0411	0.0503	0.0535	0.0503	0.0411	0.0265

Fig. 3. System 2.

a) Since the last machine is not blocked and the first is not starved, production rate can be estimated as

$$\begin{aligned} &\mathrm{PR}(p_1, r_1, p_2, r_2, \ldots, p_M, r_M, N_1, N_2, \ldots, N_{M-1}) \\ &\quad = \frac{r_M^f}{p_M^f + r_M^f} = \frac{r_1^b}{p_1^b + r_1^b}. \end{aligned} \tag{3.5}$$

Here, the first equality follows from (2.1) and the last from (3.4).

b) Estimates of machine manufacturing blockage and starvation are introduced as follows. It has been shown in [11] that the production rate of line 1)–6) with two machines can be calculated according to

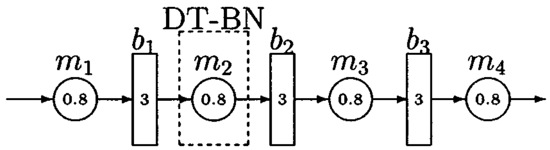
$$\begin{aligned} \tilde{\mathrm{PR}} &= e_2 - e_2 Q(p_1, r_1, p_2, r_2, N_1) \\ &= e_1 - e_1 Q(p_2, r_2, p_1, r_1, N_1) \end{aligned}$$

where e_i, $i = 1, 2$, is the machine efficiency given by (2.1) and function Q is given in (3.2). The term $e_2 Q(p_1, r_1, p_2, r_2, N_1)$ [respectively, $e_1 Q(p_2, r_2, p_1, r_1, N_1)$] is the probability of manufacturing starvation of machine m_2 (respectively, manufacturing blockage of machine m_1). Since the production rate estimate (3.5) can be rewritten as

$$\begin{aligned} &\mathrm{PR}(p_1, r_1, p_2, r_2, \ldots, p_M, r_M, N_1, N_2, \ldots, N_{M-1}) \\ &\quad = e_M - e_M Q(p_{M-1}^f, r_{M-1}^f, p_M^b, r_M^b, N_{M-1}) \\ &\quad = e_1 - e_1 Q(p_2^b, r_2^b, p_1^f, r_1^f, N_1) \end{aligned}$$

we define the probabilities of manufacturing blockage and starvation in system 1)–6) with $M > 2$ as

$$ms_i = e_i Q(p_{i-1}^f, r_{i-1}^f, p_i^b, r_i^b, N_{i-1}), \qquad i = 2, \ldots, M$$



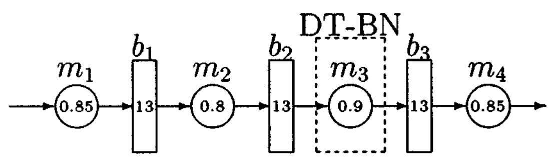
$\widetilde{PR} = 0.7267$

$PR = 0.7321$

$\frac{\widetilde{PR}-PR}{\widetilde{PR}} \cdot 100\% = 0.74\%$

	m_1		m_2		m_3		m_4
T_{up_i} :	10		5		12.5		20
T_{down_i} :	2.5		1.25		3.125		5
$\widetilde{ms}_i$:	0		0.0566		0.1053		0.1955
ms_i :	0		0.0594		0.1148		0.2005
$\lvert\widetilde{ms}_i - ms_i\rvert/\widetilde{ms}_i \cdot 100\%$:	--		4.95		9.02		2.56
$\widetilde{mb}_i$:	0.1971		0.1396		0.0909		0
mb_i :	0.2005		0.1523		0.1001		0
$\lvert\widetilde{mb}_i - mb_i\rvert/\widetilde{mb}_i \cdot 100\%$:	1.73		9.1		10.12		--
$\tilde{\sigma}_j$:	0	$\nearrow$	0.4284	$\swarrow$	4.6406	$\swarrow$	19.5543
$\tilde{\beta}_i$:	4.9273		0.9391		4.0891		0
σ_i :	0	$\nearrow$	0.459	$\swarrow$	5.1244	$\swarrow$	20.0475
β_i :	5.0119		1.0286		4.5629		0
$\lvert\Delta\widetilde{PR}/\Delta T_{down_i}\rvert$:	0.0307		0.061		0.0297		0.0192
$\lvert\Delta PR/\Delta T_{down_i}\rvert$:	0.0254		0.0696		0.0308		0.0166

Fig. 4. System 3.



$\widetilde{PR} = 0.6045$

$PR = 0.5995$

$\frac{\widetilde{PR}-PR}{\widetilde{PR}} \cdot 100\% = 0.83\%$

	m_1		m_2		m_3		m_4
T_{up_i} :	66.667		50		20		66.667
T_{down_i} :	11.765		12.5		2.222		11.765
$\widetilde{ms}_i$:	0		0.0567		0.1485		0.1525
ms_i :	0		0.0611		0.1563		0.1504
$\lvert\widetilde{ms}_i - ms_i\rvert/\widetilde{ms}_i \cdot 100\%$:	--		7.76		5.25		1.38
$\widetilde{mb}_i$:	0.1531		0.0475		0.0558		0
mb_i :	0.1504		0.0425		0.0534		0
$\lvert\widetilde{mb}_i - mb_i\rvert/\widetilde{mb}_i \cdot 100\%$:	1.76		10.53		4.3		--
$\tilde{\sigma}_j$:	0	$\nearrow$	37.686	$\nearrow$	7.0368	$\swarrow$	119.61
$\tilde{\beta}_i$:	120.05		31.97		2.971		0
σ_i :	0	$\nearrow$	40.3405	$\nearrow$	7.384	$\swarrow$	117.96
β_i :	117.96		28.79		2.8715		0
$\lvert\Delta\widetilde{PR}/\Delta T_{down_i}\rvert$:	0.0059		0.0093		0.0124		0.0052
$\lvert\Delta PR/\Delta T_{down_i}\rvert$:	0.0063		0.0094		0.019		0.0043

Fig. 5. System 4.

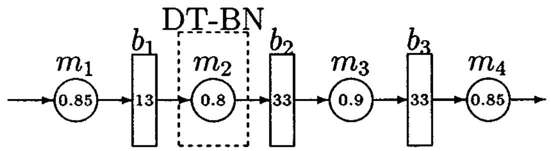
$$mb_i = e_i Q(p_{i+1}^b, r_{i+1}^b, p_i^f, r_i^f, N_i), \qquad i = 1, \ldots, M-1. \tag{3.6}$$

The accuracy of estimates (3.5) and (3.6) is discussed next.

C. Accuracy of the Estimates

The analytical investigation of estimates (3.5) and (3.6) is described in [15]. Under a number of hypotheses, it has been shown that they result in sufficiently high precision. However, the hypotheses involved are difficult to verify. Therefore, we present here only the numerical evaluation.

We simulated dozens of systems defined by assumptions 1)–6) with various machine and buffer parameters assumed. Seven of them, each illustrating a particular feature of the recursive procedure (3.1) and estimates (3.5) and (3.6), are shown in Figs. 2–8. In each simulation run, zero initial occupancy of all buffers has been assumed and 10 000 time units of warm-up period have been carried out. The time unit was assumed to be the maximum of the machines' average downtime divided by 200. The next 40 000 time units have been used to statistically estimate the production rate $\widetilde{PR}$ and the probabilities of manufacturing starvation and blockage $\widetilde{ms}_i$ and $\widetilde{mb}_i$ of each machine. Along with these "measured" performance characteristics, we calculated their analytical estimates (3.5), (3.6) and their accuracy (percentage of error). These data are shown in Figs. 2–8 along with the parameters of machines and buffers. The numbers in the circles and rectangles represent the



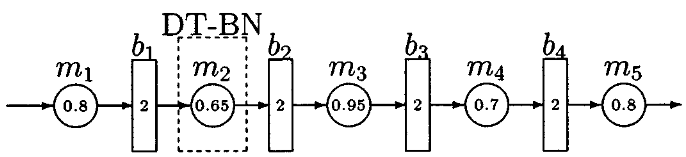
$\widetilde{PR} = 0.6959$

$PR = 0.6996$

$\frac{\widetilde{PR} - PR}{\widetilde{PR}} \cdot 100\% = 0.54\%$

T_{up_i} :	66.667	50	20	66.667
T_{down_i} :	11.765	12.5	2.222	11.765
$\widetilde{ms}_i$:	0	0.0638	0.1475	0.1216
ms_i :	0	0.0642	0.1524	0.1179
$\lvert\widetilde{ms}_i - ms_i\rvert/\widetilde{ms}_i \cdot 100\%$:	--	0.63	3.32	3.04
$\widetilde{mb}_i$:	0.1222	0.0096	0.026	0
mb_i :	0.1179	0.004	0.0187	0
$\lvert\widetilde{mb}_i - mb_i\rvert/\widetilde{mb}_i \cdot 100\%$:	3.52	58.33	28.08	--
$\tilde{\sigma}_j$:	0	40.344	6.7513	95.410
β_i :	95.860	6.5295	1.3802	0
σ_i :	0	40.3409	6.9152	92.474
β_i :	92.474	2.7206	1.0014	0
$\lvert\Delta\widetilde{PR}/\Delta T_{down_i}\rvert$:	0.0067	0.0099	0.0036	0.0025
$\lvert\Delta PR/\Delta T_{down_i}\rvert$:	0.0075	0.0105	0.0056	0.0007

Fig. 6. System 5.



$\widetilde{PR} = 0.5378$

$PR = 0.4866$

$\frac{\widetilde{PR} - PR}{\widetilde{PR}} \cdot 100\% = 9.52\%$

T_{up_i} :	10	2.857	20	3.333	5
T_{down_i} :	2.5	1.538	1.053	1.429	1.25
$\widetilde{ms}_i$:	0	0.0545	0.2588	0.1282	0.2621
ms_i :	0	0.051	0.3056	0.1883	0.3134
$\lvert\widetilde{ms}_i - ms_i\rvert/\widetilde{ms}_i \cdot 100\%$:	--	6.422	18.08	46.88	19.57
$\widetilde{mb}_i$:	0.2614	0.059	0.1537	0.0329	0
mb_i :	0.3134	0.1219	0.2325	0.0342	0
$\lvert\widetilde{mb}_i - mb_i\rvert/\widetilde{mb}_i \cdot 100\%$:	19.89	106.61	51.27	3.95	--
$\tilde{\sigma}_j$:	0	0.2633	6.4998	0.6408	1.6383
β_i :	6.5349	0.2833	4.4464	0.1919	0
σ_i :	0	0.276	8.5198	0.9429	1.9585
β_i :	7.8338	0.5815	7.2171	0.223	0
$\tilde{S}_i(S_i)$:	--	12.49(15.50)	--	5.25(8.01)	--
$\lvert\Delta\widetilde{PR}/\Delta T_{down_i}\rvert$:	0.027	0.0864	0.0037	0.0644	0.0307
$\lvert\Delta PR/\Delta T_{down_i}\rvert$:	0.0207	0.0828	0.018	0.0706	0.0282

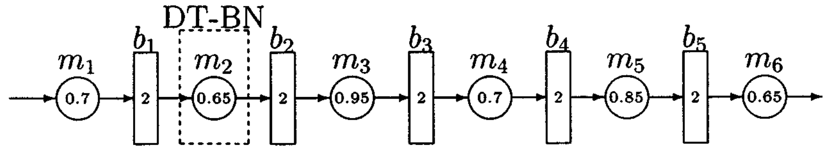
Fig. 7. System 6.

machine efficiency and buffer capacity, respectively. The last six rows of numbers in Figs. 2–8 refer to the BN identification tool, considered in Section IV, and will be commented upon later on.

The production line in Fig. 2 consists of an even number of identical machines and identical buffers. In Fig. 3, a similar line but with an odd number of machines is considered. The accuracy of production rate evaluation in both system is within 5% but ms and mb are estimated with much lower accuracy (about 25%). In Fig. 4, machines with equal efficiency but different uptime and downtime are illustrated. Here, the accuracy is about 1% for PR and 10% for ms and mb.

Figs. 5 and 6 represent systems, which differ only by the capacity of the buffers. The PR is evaluated in both cases with less than 1% error. For ms and mb, the accuracy is about 10% for shorter buffers and up to 60% for longer ones.

Systems in Figs. 7 and 8 represent situations where a highly reliable machine is surrounded by machines with a substantially lower efficiency. In these situations, the accuracy is consistently the lowest: about 10% for PR and over 100% for ms and mb.



$\widetilde{PR} = 0.4877$

$PR = 0.435$

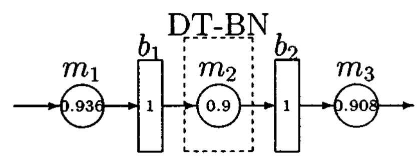
$\frac{\widetilde{PR}-PR}{PR} \cdot 100\% = 10.8\%$

T_{up_i} :	3.333	2.857	20	6.667	6.667	2.857
T_{down_i} :	1.429	1.538	1.053	2.857	1.176	1.538
$\widetilde{ms}_i$:	0	0.0637	0.2346	0.1231	0.242	0.1638
ms_i :	0	0.0589	0.3006	0.1851	0.3256	0.215
$\lvert\widetilde{ms}_i - ms_i\rvert/\widetilde{ms}_i \cdot 100\%$:	--	7.53	28.13	50.37	34.55	31.26
$\widetilde{mb}_i$:	0.2119	0.0994	0.2281	0.0907	0.1205	0
mb_i :	0.265	0.1717	0.3137	0.1086	0.1448	0
$\lvert\widetilde{mb}_i - mb_i\rvert/\widetilde{mb}_i \cdot 100\%$:	25.06	72.74	37.53	19.74	20.17	--
$\tilde{\sigma}_j$:	0	0.3303	6.5	2.6926	2.2116	0.72
β_i :	1.0092	0.4842	6.3765	2.0953	1.3212	0
σ_i :	0	0.3517	9.4469	4.1737	3.0786	0.945
β_i :	1.2619	0.8297	9.6595	2.8118	1.8415	0
$\tilde{S}_i(S_i)$:	--	6.69(9.53)	--	3.8(5.75)	--	0.60(0.90)
$\lvert\Delta\widetilde{PR}/\Delta T_{down_i}\rvert$:	0.0461	0.0612	0.0093	0.0328	0.0246	0.0478
$\lvert\Delta PR/\Delta T_{down_i}\rvert$:	0.0338	0.0608	0.0134	0.0287	0.0298	0.0398

Fig. 8. System 7.

A few remarks concerning these data are in order.

1) In all cases considered, PR is evaluated with a higher precision than ms and mb. Apparently, this happens because the absolute value of the latter is much smaller than that of the former, and the error of a similar (or even smaller) absolute value results in a higher percentage.
2) Even though PR is evaluated, in some cases, with a substantial error, given that the data available on the factory floor concerning machine and buffer parameters, are typically very unreliable and rarely are within 10% of their real values, even these nonprecise estimates are of practical importance.
3) The probabilities of manufacturing starvation and blockage ms and mb are often evaluated by (3.5) and (3.6) with a very large percent of error. However, since the absolute values of the errors are quite small, these estimates work well for BN machine identification, which relies on relative rather than absolute values of ms and mb (see Section IV).
4) Contrary to the popular belief, increasing buffer capacity does not necessarily lead to an increase in the accuracy of the performance measures estimates.
5) The accuracy of the estimates is always lower when a reliable machine is surrounded by unreliable ones. Apparently, this happens because the reliable machine introduces strong interactions between the machines, and a weak coupling, assumed *de facto* by recursive procedure (3.1), is no longer valid.
6) Recursive procedure (3.1) converges very fast: the time necessary to evaluate (3.1)–(3.6) is a fraction of a second (using a Pentium 133-MHz processor). In contrast, discrete events simulations to estimate $\widetilde{\mathrm{PR}}$, $\widetilde{\mathrm{ms}}$, and $\widetilde{\mathrm{mb}}$ are hours in duration.



T_{up_i} :	7.3	10	11
T_{down_i} :	0.5	1.1111	1.1111
$\tilde{\sigma}_j$:	0	0.2497	0.9756
β_i :	0.3885	0.554	0
σ_i :	0	0.2956	1.0285
β_i :	0.408	0.5976	0
$\lvert\Delta\widetilde{PR}/\Delta T_{down_i}\rvert$:	0.0844	0.0726	0.0634
$\lvert\Delta PR/\Delta T_{down_i}\rvert$:	0.0781	0.0693	0.0597

Fig. 9. Counterexample 1.

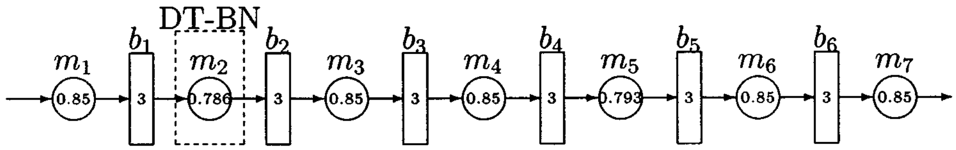
Estimates (3.5) and (3.6) are used below to introduce and evaluate a DT-BN identification tool for production lines 1)–6).

IV. DT-BN Identification Tool

It was shown in [11] that machine m_2 is the DT-BN of a two-machine line if

$$mb_1 T_{\mathrm{up}_1} T_{\mathrm{down}_1} > ms_2 T_{\mathrm{up}_2} T_{\mathrm{down}_2}. \tag{4.1}$$

When the inequality is reserved, machine m_1 is the DT-BN. Unfortunately, criterion (4.1) cannot be directly extended to M-machine lines. The reason is in the following. Consider a two-machine Markovian line and calculate ms_2. Attach the third machine to the line. In this case, ms_2 will decrease (since m_2 may now be blocked and its demand for parts will decrease). To account for this possibility and to compensate for the reduction of ms_2, one may introduce an empirical



	m_1		m_2		m_3		m_4		m_5		m_6		m_7
T_{up_i} :	17		11		17		17		11.5		17		17
T_{down_i} :	3		3		3		3		3		3		3
$\tilde{\sigma}_i$:	0	↗	1.76	↙	7.14	↗	8.14	↗	5.51	↙	12.33	↙	13.90
$\tilde{\beta}_i$:	13.91		5.82		9.45		8.35		2.99		3.17		0
σ_i :	0	↗	2.27	↙	9.70	↗	11.53	↗	7.46	↙	15.23	↙	16.24
β_i :	16.24		7.68		12.93		11.40		4.22		4.62		0
$\tilde{S}_i(S_i)$:	--		13.47(15.98)		--		--		12.18(14.95)		--		--
$\lvert\Delta\widetilde{PR}/\Delta T_{down_i}\rvert$:	0.0142		0.0246		0.0195		0.020		0.0250		0.0150		0.0108
$\lvert\Delta PR/\Delta T_{down_i}\rvert$:	0.0116		0.0235		0.0185		0.0190		0.0249		0.0152		0.0112

Fig. 10. Counterexample 2.

coefficient $1/(1 - \mathrm{mb}_2/e_2)$. When mb_2 is very small, this coefficient is, obviously, close to 1. However, when mb_2 is significant, this coefficient modifies the measured or calculated ms_2 to produce an estimate of machine m_2 starvation if no blockage would have taken place (i.e., the starvation of m_2 rather m_2^f).

Based on the above arguments, introduce the following parameters:

$$\begin{aligned}\beta_i &= \frac{mb_i}{1 - ms_i/e_i} T_{\mathrm{up}_i} T_{\mathrm{down}_i}, \qquad i = 1, \ldots, M-1\\ \sigma_i &= \frac{ms_i}{1 - mb_i/e_i} T_{\mathrm{up}_i} T_{\mathrm{down}_i} \qquad i = 2, \ldots, M.\end{aligned} \tag{4.2}$$

Using these parameters, we formulate the DT-BN identification criterion for long Markovian lines.

Consider M-machine Markovian lines shown in Figs. 2–8. Assign arrows directed from one machine to another according to the following.

Rule 4.1: If

$$\beta_j > \sigma_{j+1}, \qquad j = 1, \ldots, M-1$$

the arrow is directed from machine j to machine $j+1$. If

$$\beta_j < \sigma_{j+1}, \qquad j = 1, \ldots, M-1$$

the direction of the arrow is reversed.

Finally, introduce the numbers S_i defined as follows:

$$\begin{aligned}S_1 &= \sigma_2 - \beta_1\\ S_i &= (\beta_{i-1} + \sigma_{i+1}) - (\beta_i + \sigma_i), \qquad i = 2, \ldots, M-1\\ S_M &= \beta_{M-1} - \sigma_M.\end{aligned} \tag{4.3}$$

We refer to these numbers as *BN severity*.

DT-BN Indicator 4.1: Consider a serial production line with the arrows assigned according to Rule 4.1. Then, if there is a single machine with no arrows emanating from it, this machine is the DT-BN. If there are multiple machines with no emanating arrows, the one with the largest severity is the DT-BN.

This criterion was justified numerically. Several examples are illustrated in Figs. 2–8. In these figures, along with the data com-

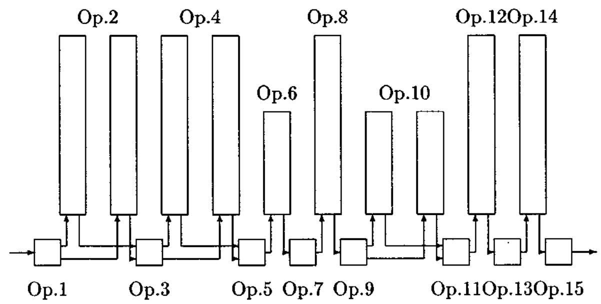


Fig. 11. Topology of the production line.

mented on in Section III, the values of $\tilde{\sigma}_i$, $\tilde{\beta}_i$ and σ_i, β_i are included. Both sets are calculated using (4.2), based on the "measured" in discrete event simulations $\tilde{\mathrm{ms}}_i$ and $\tilde{\mathrm{mb}}_i$ for the former and (3.5) and (3.6) ms_i and mb_i for the latter. In addition, estimates of sensitivity $\partial\mathrm{PR}/\partial T_{\mathrm{down}_i}$ are included. Both "measured" and analytically estimated sensitivity, $\Delta\tilde{\mathrm{PR}}/\Delta T_{\mathrm{down}_i}$ and $\Delta\mathrm{PR}/\Delta T_{\mathrm{down}_i}$ have been evaluated by finite differences with step $\Delta T_{\mathrm{down}_i} = 0.1 T_{\mathrm{down}_i}$.

As it follows from these figures, Indicator 4.1 correctly identifies the DT-BN machine. A minor discrepancy is in Fig. 2, where, according to the simulations, m_3 is the DT-BN whereas Indicator 4.1 shows that both m_3 and m_4 are the DT-BNs. We attribute this discrepancy to deficiency of simulations, since, as it follows from the reversibility property of serial lines [16], $\partial\tilde{\mathrm{PR}}/\partial T_{\mathrm{down}_3}$ should be equal to $\partial\tilde{\mathrm{PR}}/\partial T_{\mathrm{down}_4}$, as well as $\tilde{\mathrm{ms}}_3 = \tilde{\mathrm{mb}}_4$ and $\tilde{\mathrm{mb}}_3 = \tilde{\mathrm{ms}}_4$.

In Fig. 3, the middle machine m_4 is the DT-BN, as it is expected to be in accordance with the so-called bowl phenomenon [17], [18]. In Fig. 4, the machine with the shortest uptime and downtime is the DT-BN, again as it is expected in view of the results of [11]. In Fig. 5, the machine with the highest efficiency m_3 is the DT-BN, illustrating that the machine with the lowest efficiency is not necessarily the BN. In Fig. 6, where the above DT-BN is protected by larger buffers, the DT-BN shifts to the worst machine m_2. Finally, in Figs. 7 and 8, systems with several machines having no emanating arrows are illustrated (the values of BN severity $\tilde{S}$ and S, calculated according to (4.3), are also included here). Again, DT-BNs are identified correctly.

TABLE I
OPERATIONS PARAMETERS

parameters	Op.1	Op.2	Op.3	Op.4	Op.5	Op.6	Op.7	Op.8
Cycle time [sec] :	3	3	3	3	3	3	3	3
up-time ratio:	.994	.99	.997	.992	.994	.996	.998	.996
Avg. down-time [min]:	3.0	0.44	3.0	0.4	3.0	1.0	3.0	0.4
Isolation PR[parts/hr]:	1193	1188	1196	1190	1193	1195	1198	1195
	Op.9	Op.10	Op.11	Op.12	Op.13	Op.14	Op.15	–
Cycle time [sec] :	3	3	3	3	3	3	3	
up-time ratio:	.994	.992	.994	.996	.994	.998	.997	
Avg. down-time [min]:	3.0	0.5	3.0	0.57	3.0	0.33	3.0	
Isolation PR[parts/hr]:	1193	1190	1193	1195	1193	1198	1196	

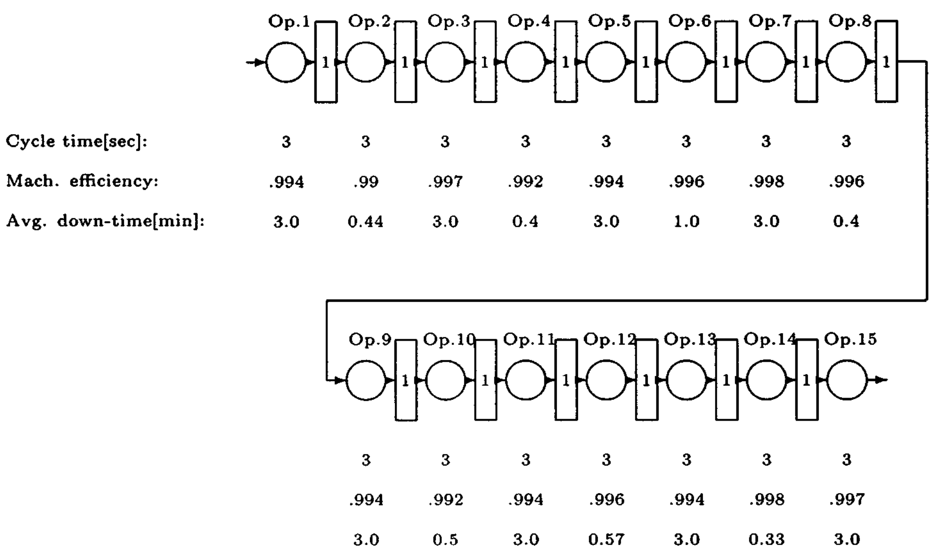


Fig. 12. Model of the production system.

Along with the above, a few counterexamples to Indicator 4.1 have been discovered. Two of them are shown in Figs. 9 and 10. In each counterexample, however, the difference between the sensitivity of the production rate to the downtime of the actual DT-BN and the one identified using Indicator 4.1 is very small. Therefore, we conclude that Indicator 4.1 can be used as a tool for DT-BN identification in Markovian serial production lines.

An application of this Indicator in a case study is discussed next.

V. Case Study

The DT-BN indicator described above was used in the design of a production line at an automotive component plant. The layout of the system is shown in Fig. 11. It consists of 15 operations (machines) each performing a complex automated calibration, test, or transfer manipulation. Performance and reliability characteristics of the machines are summarized in Table I. According to these data, Op. 2 is the worst machine in the system, as far as the production rate in isolation is concerned. The problem was to determine if Op. 2 is indeed the BN and, if not, suggest measures for achieving this situation and increasing production rate of the system as a whole.

A model of the production line at hand with the machine and buffer parameters identified from Table I and from the system layout is shown in Fig. 12. Using recursive procedure

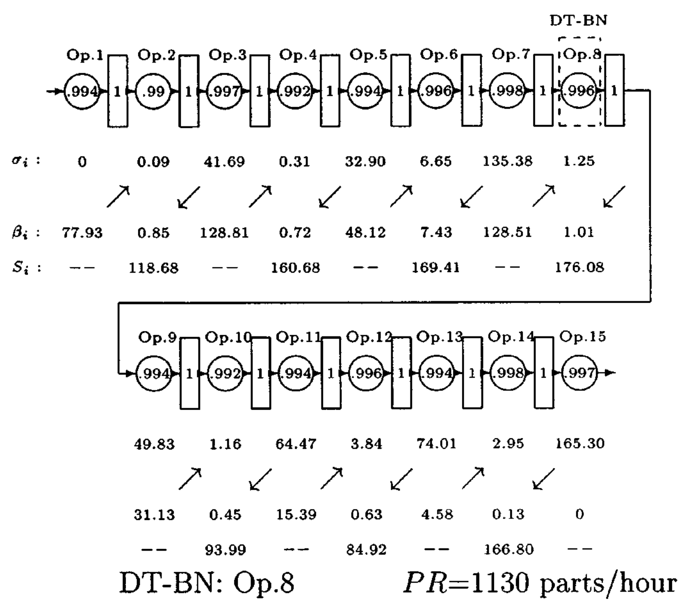


Fig. 13. DT-BN identification: the original design.

(3.1)–(3.6) and DT-BN indicator of Section IV, we arrive at Fig. 13, which indicates that the DT-BN is Op. 8. Since Op. 8 is not the worst in the system, we conclude that the buffer structure proposed is not potent *viz-a-viz* the predicted machine

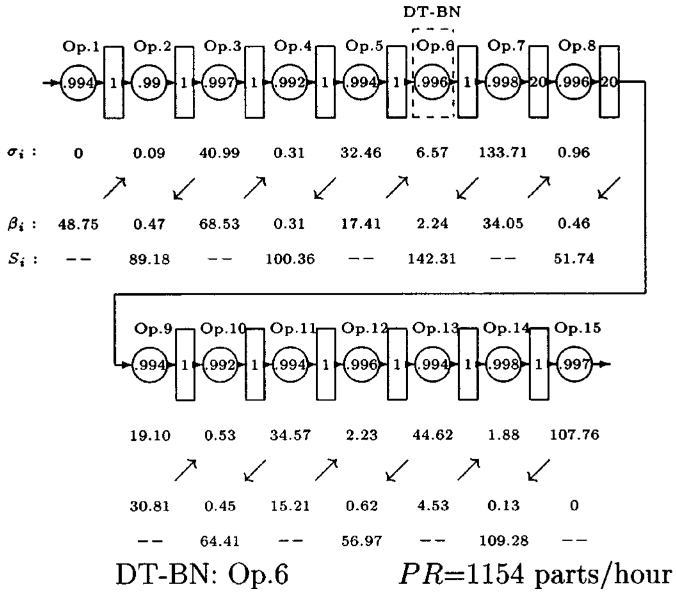


Fig. 14. DT-BN identification: $N_i = 20, i = 7, 8$.

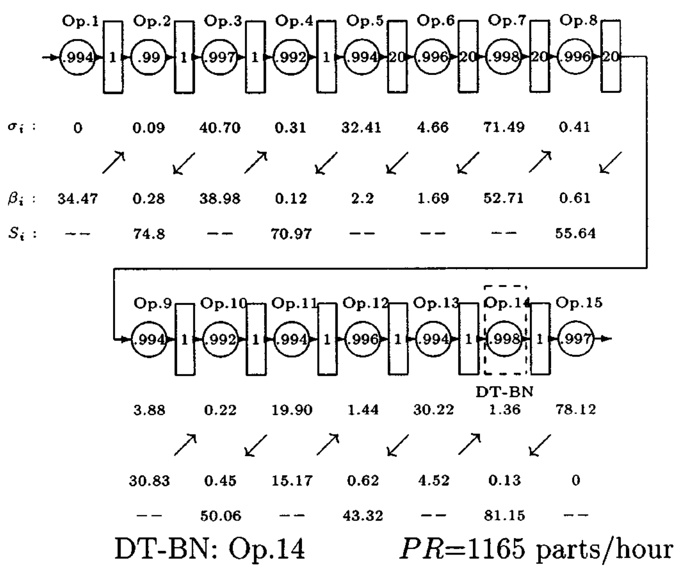


Fig. 15. DT-BN identification: $N_i = 20, i = 5, 6, 7, 8$.

reliability, and an improvement of the performance may be obtained by modifying the capacity of the buffers.

To protect the DT-BN, we increase the capacity of the buffers in front and after Op. 8 from 1 to 20. The resulting performance is shown in Fig. 14; DT-BN now is Op. 6. Increasing b_5 and b_6 to 20, we calculate that the new DT-BN is Op. 14 (Fig. 15). Protecting this operation by larger buffers, we arrive at the system in Fig. 16, where the DT-BN is Op. 2. Since this is the slowest machine in the system and since the production rate (1166 parts/h) is close to that of Op. 2 in isolation (1188 parts/h), we conclude that the buffer structure arrived at is potent.

The plant management incorporated these recommendations into the final design of the system. At present, the system is fully operational and, according to production personnel, operates at the level predicted by this study. Unfortunately, no quantitative information to this effect could be obtained from the plant.

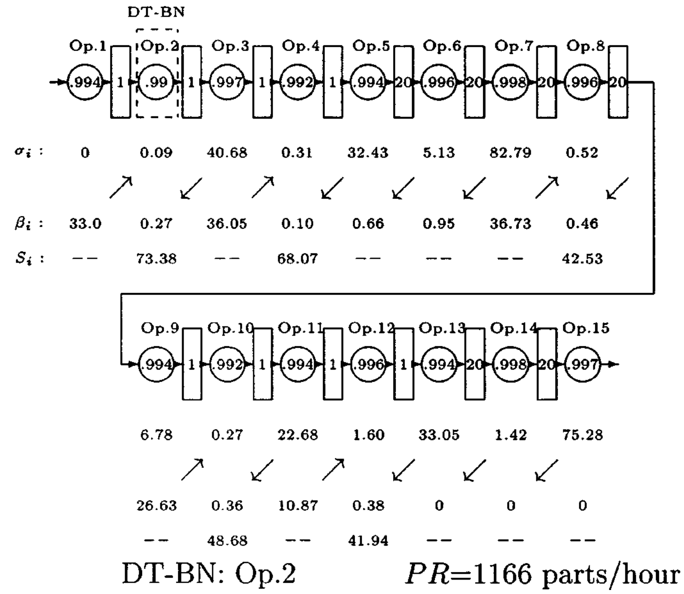


Fig. 16. DT-BN identification: $N_i = 20, i = 5, 6, 7, 8, 13, 14$.

VI. Conclusions

This paper presents a simple method for DT-BN identification in serial production lines with Markovian machines (i.e., when uptime and downtime are distributed exponentially). The information necessary for its application is the average uptime and downtime of the machines and the frequency of their blockages and starvations. Both can be measured on the factory floor during the normal system operation. Alternatively, they can be evaluated using the performance analysis technique described in Section III. Thus, the results of this paper provide a practical tool for BN identification in large volume manufacturing environment.

Appendix
Proofs for Section III

To prove Theorem 3.1, we need the following facts.

Lemma A.1: Function $Q(p_x, r_x, p_y, r_y, N)$, $N \in R_+$, defined in (3.2), has the following properties:

a) monotonically increasing in p_x;
b) monotonically decreasing in r_x;
c) monotonically decreasing in p_y;
d) monotonically increasing in r_y;
e) monotonically decreasing in N;
f) takes values in (0,1).

Proof: For the case $e_x \neq e_y$, where $e_x = r_x/(p_x+r_x)$ and $e_y = r_y/(p_y + r_y)$, rewrite $Q(p_x, r_x, p_y, r_y, N)$ as follows:

$$Q(p_x, r_x, p_y, r_y, N) = \frac{1 - e_x}{1 + \dfrac{\phi}{1-\phi}(1 - e^{-\beta N})} \tag{A.1}$$

where $\phi = (e_x(1-e_y))/(e_y(1-e_x))$ and $\beta = ((r_x + r_y + p_x + p_y)(p_x r_y - p_y r_x))/((r_x + r_y)(p_x + p_y))$. Differentiating the denominator of (A.1) with respect to p_x, we obtain (A.2), shown at the bottom of the page. Consider two cases: $e_x < e_y$ and $e_x > e_y$. If $e_x < e_y$, we have

$$p_x r_y > p_y r_x \quad \text{and} \quad \beta > 0. \tag{A.3}$$

Introduce $f(N)$ by (A.4), shown at the bottom of the page. Since $f(0) = r_x p_y r_y/((p_x r_y - p_y r_x)^2)$, $f(\infty) = 0$, and

$$\begin{aligned}\frac{\partial f(N)}{\partial N} &= e^{-\beta N}\left\{\frac{-p_y r_x}{(p_x+p_y)^2}\right.\\ &\quad \left.-\beta N \frac{p_y r_x[(p_x+p_y)^2 r_y + (r_x+r_y)^2 p_y]}{(p_x+p_y)^2(r_x+r_y)(p_x r_y - p_y r_x)}\right\}\\ &< 0\end{aligned} \tag{A.5}$$

function $f(N)$ takes values in $[0, (r_x p_y r_y/(p_x r_y - p_y r_x)^2)]$. Substituting the value of $f(N)$ into (A.2), we obtain

$$\frac{\partial\left[1 + \dfrac{\phi}{1-\phi}(1 - e^{-\beta N})\right]}{\partial p_x} \leq 0 \tag{A.6}$$

i.e., the denominator of (A.1) is monotonically decreasing in p_x. Therefore, using (A.6) and the fact that the numerator of (A.1) is monotonically increasing in p_x

$$\frac{\partial Q(p_x, r_x, p_y, r_y, N)}{\partial p_x} > 0. \tag{A.7}$$

If $e_x > e_y$, we have

$$p_x r_y < p_y r_x \quad \text{and} \quad \beta < 0. \tag{A.8}$$

From (A.4), (A.5), and (A.8)

$$f(N) \in \left(-\infty, \frac{r_x p_y r_y}{(p_x r_y - p_y r_x)^2}\right]. \tag{A.9}$$

Hence, using (A.2)

$$\frac{\partial\left[1 + \dfrac{\phi}{1-\phi}\left(1 - e^{-\beta N}\right)\right]}{\partial p_x} \leq 0. \tag{A.10}$$

Similarly, statement a) holds for the case $e_x > e_y$ as well.

If $e_x = e_y$, calculating $Q(p_x + \Delta p, r_x, p_y, r_y, N)$ yields

$$Q(p_x + \Delta p, r_x, p_y, r_y, N) = \frac{(1 - e'_x)(1 - \phi')}{1 - \phi' e^{-\beta N}}$$

where $\Delta p > 0$, $\phi' = (e'_x(1 - e_y))/(e_y(1 - e'_x))$, $e'_x = r_x/(p_x + \Delta p + r_x)$, and $e_y = r_y/(p_y + r_y)$. Thus, $Q(p_x + \Delta p, r_x, p_y, r_y, N)$ is defined in the first equation shown at the bottom of the next page. Therefore, we get the middle equation shown at the bottom of the next page, i.e., $Q(p_x, r_x, p_y, r_y, N)$ is monotonically increasing in p_x for the case $e_x = e_y$. Similarly, we obtain

$$\lim_{\Delta p \to 0} \frac{(Q(p_x, r_x, p_y, r_y, N) - Q(p_x - \Delta p, r_x, p_y, r_y, N))}{\Delta p} > 0.$$

This proves statement a). Statements b), c), and d) are proved analogously.

For statements e) and f), if $e_x < e_y$, from (A.1) and (A.3)

$$\phi < 1 \qquad \beta > 0.$$

Thus, the term $(\phi/(1-\phi))(1 - e^{-\beta N})$ in the denominator of (A.1) is positive and monotonically increasing in N. Therefore, we conclude that $0 < Q(p_x, r_x, p_y, r_y, N) < 1$ and $Q(p_x, r_x, p_y, r_y, N)$ is monotonically decreasing in N.

If $e_x > e_y$, from (A.1) and (A.8)

$$\phi > 1 \qquad \beta < 0$$

which imply $(\phi/(1-\phi))(1 - e^{-\beta N})$ in (A.1) is positive and monotonically increasing in N. Hence, $0 < Q(p_x, r_x, p_y, r_y, N) < 1$ and $Q(p_x, r_x, p_y, r_y, N)$ is monotonically decreasing in N.

Consider the case of $e_x = e_y$. Since $N \geq 0$, using (3.1), we get the equation shown at the bottom of the page 13. From this expression, it is easy to see that function $Q(p_x, r_x, p_y, r_y, N)$ is monotonically decreasing in N. This completes the proof. ■

Lemma A.2: Consider $p_i^f(s)$, $r_i^f(s)$, $p_i^b(s)$, and $r_i^b(s)$, $i = 1, \ldots, M$, defined by recursive procedure (3.1). If for all $j = 2, \ldots, M$, $p_j^f(s) > p_j^f(s-1)$ and $r_j^f(s) < r_j^f(s-1)$, then for all $j = 1, \ldots, M-1$, $p_j^b(s+1) < p_j^b(s)$, and $r_j^b(s+1) > r_j^b(s)$.

$$\begin{aligned}\frac{\partial}{\partial p_x}\left[1 + \frac{\phi}{1-\phi}(1 - e^{-\beta N})\right] &= \frac{-r_x p_y r_y}{(p_x r_y - p_y r_x)^2} + e^{-\beta N}\\ &\quad \cdot \left\{\frac{r_x p_y r_y}{(p_x r_y - p_y r_x)^2} + N\frac{r_x p_y[(p_x+p_y)(r_x+r_y+p_x+p_y)r_y - (r_x+r_y)(p_x r_y - p_y r_x)]}{(p_x+p_y)^2(r_x+r_y)(p_x r_y - p_y r_x)}\right\}\end{aligned} \tag{A.2}$$

$$e^{-\beta N}\left\{\frac{r_x p_y r_y}{(p_x r_y - p_y r_x)^2} + N\frac{r_x p_y[(p_x+p_y)(r_x+r_y+p_x+p_y)r_y - (r_x+r_y)(p_x r_y - p_y r_x)]}{(p_x+p_y)^2(r_x+r_y)(p_x r_y - p_y r_x)}\right\} = f(N) \tag{A.4}$$

Proof (By Induction): Under the above assumption, for $j = M-1$, using Lemma A.1, we obtain

$$\begin{aligned}&Q(p_M,\, r_M,\, p^f_{M-1}(s),\, r^f_{M-1}(s),\, N_{M-1})\\ &\quad < Q(p_M,\, r_M,\, p^f_{M-1}(s-1),\, r^f_{M-1}(s-1),\, N_{M-1}).\end{aligned}$$

From the recursive procedure (3.1), due to the above inequality

$$\begin{aligned}&p^b_{M-1}(s+1)\\ &\quad = \frac{p_{M-1}}{1 - Q(p_M, r_M, p^f_{M-1}(s), r^f_{M-1}(s), N_{M-1})}\\ &\quad < \frac{p_{M-1}}{1 - Q(p_M, r_M, p^f_{M-1}(s-1), r^f_{M-1}(s-1), N_{M-1})}\\ &\quad = p^b_{M-1}(s)\\ &r^b_{M-1}(s+1)\\ &\quad = \frac{1}{\dfrac{Q(p_M, r_M, p^f_{M-1}(s), r^f_{M-1}(s), N_{M-1})}{p_{M-1}} + \dfrac{1}{r_{M-1}}}\\ &\quad > \frac{1}{\dfrac{Q(p_M, r_M, p^f_{M-1}(s-1), r^f_{M-1}(s-1), N_{M-1})}{p_{M-1}} + \dfrac{1}{r_{M-1}}}\\ &\quad = r^b_{M-1}(s).\end{aligned}$$

For $j = M-2,\, M-3,\, \ldots 2,\, 1$, we have

$$\begin{aligned}&Q(p^b_{j+1}(s+1),\, r^b_{j+1}(s+1),\, p^f_j(s),\, r^f_j(s),\, N_j)\\ &\quad < Q(p^b_{j+1}(s),\, r^b_{j+1}(s),\, p^f_j(s),\, r^f_j(s),\, N_j)\\ &\quad < Q(p^b_{j+1}(s),\, r^b_{j+1}(s),\, p^f_j(s-1),\, r^f_j(s-1),\, N_j).\end{aligned}$$

Therefore

$$\begin{aligned}&p^b_j(s+1)\\ &\quad = \frac{p_j}{1 - Q(p^b_{j+1}(s+1), r^b_{j+1}(s+1), p^f_j(s), r^f_j(s), N_j)}\\ &\quad < \frac{p_j}{1 - Q(p^b_{j+1}(s), r^b_{j+1}(s), p^f_j(s-1), r^f_j(s-1), N_j)}\\ &\quad = p^b_j(s),\\ &r^b_j(s+1)\\ &\quad = \frac{1}{\dfrac{Q(p^b_{j+1}(s+1), r^b_{j+1}(s+1), p^f_j(s), r^f_j(s), N_j)}{p_j} + \dfrac{1}{r_j}}\\ &\quad > \frac{1}{\dfrac{Q(p^b_{j+1}(s), r^b_{j+1}(s), p^f_j(s-1), r^f_j(s-1), N_j)}{p_j} + \dfrac{1}{r_j}}\\ &\quad = r^b_j(s).\end{aligned}$$

■

$$\begin{aligned}&Q(p_x + \Delta p,\, r_x,\, p_y,\, r_y,\, N)\\ &\quad = \frac{\dfrac{p_x + \Delta p}{p_x + \Delta p + r_x}\,\dfrac{\Delta p}{p_x + \Delta p}}{1 - \dfrac{r_x p_y}{r_y(p_x + \Delta p)} e^{-(((r_x + r_y + p_x + p_y) r_y \Delta p)/((p_x + \Delta p + p_y)(r_x + r_y)))N}}\\ &\quad = \frac{\dfrac{p_x + \Delta p}{p_x + \Delta p + r_x}\,\dfrac{\Delta p}{p_x + \Delta p}}{1 - \dfrac{r_x p_y}{r_y(p_x + \Delta p)}\left[1 - \dfrac{(r_x + r_y + p_x + p_y) r_y \Delta p}{(p_x + \Delta p + p_y)(r_x + r_y)} N\right]} + \mathcal{O}(\Delta p^2)\\ &\quad = \frac{(p_x + p_y)(r_x + r_y)(p_x + \Delta p)}{(p_x + \Delta p + r_x)[(p_x + p_y)(r_x + r_y) + r_x p_y (r_x + r_y + p_x + p_y) N]} + \mathcal{O}(\Delta p^2).\end{aligned}$$

$$\begin{aligned}&\lim_{\Delta p \to 0} \frac{(Q(p_x + \Delta p, r_x, p_y, r_y, N) - Q(p_x, r_x, p_y, r_y, N))}{\Delta p}\\ &\quad = \lim_{\Delta p \to 0} \left\{ \frac{\dfrac{(p_x + p_y)(r_x + r_y)(p_x + \Delta p)}{(p_x + \Delta p + r_x)[(p_x + p_y)(r_x + r_y) + r_x p_y (r_x + r_y + p_x + p_y) N]} - \dfrac{(p_x + p_y)(r_x + r_y) p_x}{(p_x + r_x)[(p_x + p_y)(r_x + r_y) + r_x p_y (r_x + r_y + p_x + p_y) N]}}{\Delta p} + \mathcal{O}(\Delta p) \right\}\\ &\quad = \frac{(p_x + p_y)(r_x + r_y) r_x}{(p_x + r_x)^2[(p_x + p_y)(r_x + r_y) + r_x p_y (r_x + r_y + p_x + p_y) N]} > 0\end{aligned}$$

Lemma A.3: If for all $j = 1, \ldots, M-1$, $p_j^b(s+1) < p_j^b(s)$ and $r_j^b(s+1) > r_j^b(s)$, then for all $j = 2, \ldots, M$, $p_j^f(s+1) > p_j^f(s)$ and $r_j^f(s+1) < r_j^f(s)$.

Proof: Similar to the proof of Lemma A.2. ■

Lemma A.4: Sequences $p_j^f(s)$ and $p_j^b(s)$ are monotonically increasing and sequences $r_j^f(s)$ and $r_j^b(s)$ are monotonically decreasing.

Proof: By Induction: For $s = 0$, due to statement f) of Lemma A.1, we have

$$\begin{aligned} p_j^f(1) &= \frac{p_j}{1 - Q(p_{j-1}^f(1), r_{j-1}^f(1), p_j^b(1), r_j^b(1), N_{j-1})} \\ &> p_j = p_j^f(0), \qquad 2 \le j \le M \\ r_j^f(1) &= \frac{1}{\dfrac{Q(p_{j-1}^f(1), r_{j-1}^f(1), p_j^b(1), r_j^b(1), N_{j-1})}{p_j} + \dfrac{1}{r_j}} \\ &< r_j = r_j^f(0), \qquad 2 \le j \le M. \end{aligned}$$

Assume that for $s > 0$

$$\begin{aligned} p_j^f(s) &> p_j^f(s-1), \qquad 2 \le j \le M \\ r_j^f(s) &< r_j^f(s-1), \qquad 2 \le j \le M. \end{aligned}$$

Then by Lemma A.2

$$\begin{aligned} p_j^b(s+1) &< p_j^b(s), \qquad 1 \le j \le M-1 \\ r_j^b(s+1) &> r_j^b(s), \qquad 1 \le j \le M-1. \end{aligned}$$

So, by Lemma A.3

$$\begin{aligned} p_j^f(s+1) &> p_j^f(s), \qquad 2 \le j \le M \\ r_j^f(s+1) &< r_j^f(s), \qquad 2 \le j \le M. \end{aligned}$$

■

Proof of Theorem 3.1: Since the sequences $p_i^f(s)$, $r_i^f(s)$, $p_i^b(s)$, and $r_i^b(s)$, $1 \le j \le M$, are monotonic (Lemma A.4) and bounded from above and below (Lemma A.1), they are convergent.

To prove that $r_M^f/p_M^f = r_1^b/p_1^b$, consider the steady-state equations of the recursive procedure (3.1)

$$\begin{aligned} p_i^b &= \frac{p_i}{1 - Q(p_{i+1}^b, r_{i+1}^b, p_i^f, r_i^f, N_i)}, \qquad 1 \le i \le M-1 \\ r_i^b &= \frac{1}{\dfrac{Q(p_{i+1}^b, r_{i+1}^b, p_i^f, r_i^f, N_i)}{p_i} + \dfrac{1}{r_i}}, \qquad 1 \le i \le M-1 \\ p_i^f &= \frac{p_i}{1 - Q(p_{i-1}^f, r_{i-1}^f, p_i^b, r_i^b, N_{i-1})}, \qquad 2 \le i \le M \\ r_i^f &= \frac{1}{\dfrac{Q(p_{i-1}^f, r_{i-1}^f, p_i^b, r_i^b, N_{i-1})}{p_i} + \dfrac{1}{r_i}}, \qquad 2 \le i \le M. \end{aligned} \tag{A.11}$$

Define

$$\begin{aligned} e_i^f &= \frac{r_i^f}{r_i^f + p_i^f}, \qquad i = 1, \ldots, M \\ e_i^b &= \frac{r_i^b}{r_i^b + p_i^b}, \qquad i = 1, \ldots, M. \end{aligned} \tag{A.12}$$

From (A.11) and (A.12), we have

$$\begin{aligned} e_i^b &= e_i[1 - Q(p_{i+1}^b, r_{i+1}^b, p_i^f, r_i^f, N_i)], \qquad 1 \le i \le M-1 \\ e_i^f &= e_i[1 - Q(p_{i-1}^f, r_{i-1}^f, p_i^b, r_i^b, N_{i-1})], \qquad 2 \le i \le M \\ e_1^f &= e_1 \\ e_M^b &= e_M \end{aligned} \tag{A.13}$$

where function Q is defined in (3.2). Introduce $(M-1)$ two-machine one-buffer production lines L_i, $i = 1, \ldots, M-1$, where the first machine has the uptime parameter p_i^f and the downtime parameter r_i^f, the second p_{i+1}^b and r_{i+1}^b, and the buffer capacity is N_i. The following properties hold. Let PR_i be the production rate of line L_i, $i = 1, \ldots, M-1$, and let $\mathrm{PR}_M = e_M^f$. Then, $\mathrm{PR}_i = e_i^f e_i^b / e_i$, $i = 1, \ldots, M$. Moreover, $\mathrm{PR}_i = \mathrm{PR}_j$, $\forall j \ne i$.

The proof of the above properties is as follows. From [11], for $1 \le i \le M-1$, we have

$$\begin{aligned} \mathrm{PR}_i &= e_i^f[1 - Q(p_{i+1}^b, r_{i+1}^b, p_i^f, r_i^f, N_i)] \\ &= e_i[1 - Q(p_{i+1}^b, r_{i+1}^b, p_i^f, r_i^f, N_i)]\frac{e_i^f}{e_i}. \end{aligned} \tag{A.14}$$

Using (A.13) results in

$$\mathrm{PR}_i = \frac{e_i^f e_i^b}{e_i}, \qquad i = 1, \ldots, M-1$$

and

$$\mathrm{PR}_M = e_M^f = \frac{e_M^f e_M}{e_M} = \frac{e_M^f e_M^b}{e_M}.$$

Therefore, we conclude that $\mathrm{PR}_i = e_i^f e_i^b / e_i$, $i = 1, \ldots, M$. In addition, from (A.13)

$$\begin{aligned} \mathrm{PR}_i &= \frac{e_i^f e_i^b}{e_i} = \frac{e_i^b}{e_i} e_i[1 - Q(p_{i-1}^f, r_{i-1}^f, p_i^b, r_i^b, N_{i-1})] \\ &= e_i^b[1 - Q(p_{i-1}^f, r_{i-1}^f, p_i^b, r_i^b, N_{i-1})] \\ &= \mathrm{PR}_{i-1}, \qquad i = 2, \ldots, M. \end{aligned}$$

As it follows from the above properties, we obtain $\mathrm{PR}_1 = e_1^f e_1^b / e_1 = e_M^f e_M^b / e_M$. Since $e_1^f = e_1$ and $e_M^b = e_M$, we obtain $\mathrm{PR}_1 = e_1^b = e_M^f$, i.e., $\mathrm{PR}_1 = r_1^b/(p_1^b + r_1^b) = r_M^f/(p_M^f + r_M^f)$. Therefore, $r_M^f/p_M^f = r_1^b/p_1^b$. ■

REFERENCES

[1] J.-T. Lim, S. M. Meerkov, and F. Top, "Homogeneous, asymptotically reliable serial production lines: Theory and a case study," *IEEE Trans. Automat. Contr.*, vol. 35, pp. 524–534, May 1990.

$$0 < Q(p_x, r_x, p_y, r_y, N) = \frac{p_x(p_x + p_y)(r_x + r_y)}{(p_x + r_x)(p_x + p_y)(r_x + r_y) + p_y r_x (p_x + r_x)(p_x + p_y + r_x + r_y)N} < 1$$

[2] D. Jacobs and S. M. Meerkov, "A system-theoretic property of serial production lines: Improvability," *Int. J. Syst. Sci.*, vol. 26, no. 4, pp. 755–785, 1995.

[3] B. A. Sevastyanov, "Influence of storage bin capacity on the average standstill time of a production line," *Theory Probab. Appl.*, vol. 7, pp. 429–438, 1962.

[4] Y. Dallery and S. B. Gershwin, "Manufacturing flow line systems: A review of model and analytical results," *Queuing Syst.*, vol. 12, pp. 3–94, 1992.

[5] N. Viswanadham and Y. Narahari, *Performance Modeling of Automated Manufacturing Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1992.

[6] J. A. Buzacott and J. G. Shanthikumar, *Stochastic Models of Manufacturing Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1993.

[7] S. B. Gershwin, *Manufacturing Systems Engineering*. Englewood Cliffs, NJ: Prentice-Hall, 1994.

[8] D. D. Yao, Ed., *Stochastic Modeling and Analysis of Manufacturing Systems*. New York: Springer-Verlag, 1994.

[9] Y. Dallery, R. David, and X.-L. Xie, "Approximate analysis of transfer lines with unreliable machines and finite buffers," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 943–953, Sept. 1989.

[10] C.-T. Kuo, J.-T. Lim, and S. M. Meerkov, "Bottlenecks in serial production lines: A system-theoretic approach," *Math. Prob. Eng.*, vol. 2, no. 3, pp. 233–276, Apr. 1996.

[11] S.-Y. Chiang, C.-T. Kuo, and S. M. Meerkov, "Bottlenecks in Markovian production lines: A systems approach," *IEEE Trans. Robot. Automat.*, vol. 14, pp. 352–359, Apr. 1998.

[12] Y. C. Ho, M. A. Eyler, and T. T. Chien, "A new approach to determining parameter sensitivity of transfer lines," *Manage. Sci.*, vol. 29, pp. 700–714, 1983.

[13] R. Suri, "Perturbation analysis: The state of the art and research issues explained via the G/G/1 queue," *Proc. IEEE*, vol. 77, pp. 114–137, 1989.

[14] X. R. Cao and Y. C. Ho, *Perturbation Analysis of Discrete Event Dynamical Systems*. Norwell, MA: Kluwer, 1991.

[15] S.-Y. Chiang, C.-T. Kuo, and S. M. Meerkov, "DT-bottlenecks in Markovian production lines: Identification and application," Univ. of Michigan, Ann Arbor, MI, Control Group Rep. 98-06, 1998.

[16] E. Muth, "The reversability property of production lines," *Manage. Sci.*, vol. 25, pp. 152–158, 1979.

[17] F. S. Hillier and K. C. So, "On the optimal design of trandem queueing systems with finite buffers," *Queueing Syst.*, vol. 21, pp. 245–266, 1995.

[18] G. Yamazaki, H. Sakasegawa, and J. G. Shantikumar, "On optimal arrangement of stations in a tandem queueing system with blocking," *Manage. Sci.*, vol. 38, pp. 137–153, 1992.

Shu-Yin Chiang received the B.S. degree from the Tatung Institute of Technology, Taipei, Taiwan, R.O.C., in 1990, and the M.S. and Ph.D. degrees from the University of Michigan, Ann Arbor, in 1994 and 1999, respectively, all in electrical engineering.

She is currently an Assistant Professor in the Department of Computer Communication at Ming-Chuan University, Taiwan, R.O.C. Her research interests include manufacturing, communication networks and wireless communication.

Chih-Tsung Kuo (M'96) received the B.S. and M.S. degrees from the Tatung Institute of Technology, Taipei, Taiwan, R.O.C., in 1984 and 1986, respectively, and the Ph.D. degree from the University of Michigan, Ann Arbor, in 1996, all in electrical engineering.

He was an Associate Professor at Tatung Institute of Technology, Taiwan, R.O.C. His research interests included manufacturing, communication networks, and automatic control systems.

Dr. Kuo passed away on September 22, 1999.

Semyon M. Meerkov (M'78–SM'83–F'90) received the M.S.E.E. degree from the Polytechnic of Kharkov, Ukraine, in 1962, and the Ph.D. degree in systems science from the Institute of Control Sciences, Moscow, Russia, in 1996.

He was with the Institute of Control Sciences until 1977. From 1979 to 1984, he was with the Department of Electrical Engineering, Illinois Institute of Technology, Chicago. Since 1984, he has been a Professor at the Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor. He has held visiting positions at UCLA (1978–1979), Stanford (1991), and Lady Davis Visiting Professorship at the Technion, Haifa, Israel (1997–1998). He is Editor-in-Chief of *Mathematical Problems in Engineering* and Associate Editor for several other journals. His research interests include systems and control with applications to production systems and communication networks.